

An Algebraic Description of the Exceptional Isogenies to Orthogonal Groups

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Introduction

The orthogonal groups of finite dimensional quadratic spaces over fields appear in many branches of mathematics. They form an infinite family of algebraic groups, indexed by the dimension of the underlying space. Moreover, if the field is algebraically closed (or more generally quadratically closed) then there is only one quadratic space of any given dimension, up to isomorphism. However, over a general field \mathbb{F} (we consider only characteristic different from 2 in this manuscript) there may be many different quadratic space of the same dimension, hence many orthogonal groups of the same dimension. One way to view these subtleties is as different \mathbb{F} -rational structures on the orthogonal group over the algebraic closure of \mathbb{F} .

In lower dimensions the orthogonal groups become isomorphic (up to finite index) with members of other families of algebraic groups. These isomorphisms, especially over \mathbb{R} , are very useful in several mathematical fields: E.g., the relation between $SO^+(2, 1)$ and $SL_2(\mathbb{R})$, between $SO^+(2, 2)$ and the product of two copies of $SL_2(\mathbb{R})$, and between $SO^+(2, 3)$ and $Sp_4(\mathbb{R})$ has far-reaching applications in the theory of modular and automorphic forms—see [B], for example. Now, these isomorphisms (or isogenies) are relatively simple to describe over a quadratically closed in dimensions up to 6. Indeed, in the case of algebraically closed fields and dimensions 4 and 6 such a description appears already in [vdW], but for more general fields that reference simply refers to the resulting groups as the \mathbb{F} -rational structures mentioned above. Section III of [D] also presents some results for the general orthogonal groups in these dimensions, but again using this descent method. Moreover, some aspects of the theory becomes simpler for the general orthogonal group. Hence in some cases our method is a true refinement of that of [D]. In addition, there is an isogeny in dimension 8 over \mathbb{R} , namely signature $(2, 6)$, which is related to a symplectic quaternionic group. This relation appears in detail in [SH]. The phenomenon of triality (see [KMRT] for more details) in the isotropic case may also be viewed as a type of an exceptional isogeny.

The special orthogonal group of a quadratic space V over \mathbb{F} comes with a natural central extension, with kernel \mathbb{F}^\times , called the Gspin group or the even

Clifford group. It has a general definition in terms of Clifford algebras, namely the set of elements of the even Clifford algebra of V conjugation by which preserves the embedding of V into the full Clifford algebra. The spin group is a subgroup of the Gspin group, which maps onto the spinor norm kernel of $SO(V)$ with kernel ± 1 . It may also be described in similar terms using the Clifford algebra. It is these groups, the spin and Gspin groups, which mainly appear in the exceptional isogenies. However, the condition of preservation of V under conjugation is not so easy to verify without delving deep into the multiplicative structure of the Clifford algebra. This makes the actual structure of the groups thus obtained less visible.

We use a different, more elementary method in order to determine the spin and Gspin groups of spaces of dimensions up to 8, thus providing the groups which are isogenous to the special orthogonal groups in these dimensions. The idea is simple: We first observe that these groups are invariant under rescaling of the space, allowing us to choose our space with some extra, useful properties. We then show how a group (which ends up being the Gspin group) acts naturally on this (rescaled) space, with kernel \mathbb{F}^\times . The surjectivity is a consequence of the Cartan–Dieudonné Theorem, since we show how all the reflections can be realized in an appropriate semi-direct product which maps to the full orthogonal group, showing that we do get the full Gspin group. Moreover, in all these cases the spinor norm, which takes values in $\mathbb{F}^\times/(\mathbb{F}^\times)^2$, factors through a map to \mathbb{F}^\times , which takes r in the kernel \mathbb{F}^\times to r^2 . This allows us to determine the spin group as the subgroup of the Gspin group which maps to 1 under this map. In some isotropic cases there are particular choices or conjugations which one may apply in order for the realization of the spin and Gspin groups becomes well-known classical groups.

In many cases, especially the isotropic ones, these groups have equivalent representations, which appear in dimensions 3 and 4 in many places in the literature. We give a general, simple construction for these equivalent representations. Dimension 6 is strongly related to the second exterior power of a 4-dimensional space. We provide the resulting equivalent representations in this case too.

The possible complexity of quadratic spaces increases with the size of the group $\mathbb{F}^\times/(\mathbb{F}^\times)^2$. It is thus expedient to see what are the results for the cases where this group is small, in particular when it is 1 or 2. In the latter case one must distinguish between two cases, the Euclidean and the quadratically finite ones, according to whether the field admits a non-split quaternion algebra or not. We give full details for these cases.

This manuscript is divided into 12 sections. In Section 1 we present the required notation for central simple algebras over fields, and Section 2 contains the definitions for the groups which we shall encounter. Section 3 is concerned with dimensions up to 3, and Section 4 deals with the 4-dimensional case. In Section 5 we examine the case of 6 dimensions and trivial discriminant, and Section 6 considers 5-dimensional quadratic spaces. Section 7 we go on to general 6-dimensional quadratic spaces, and Section 8 presents the equivalent representations which arise from the exterior square of 4-dimensional spaces. Section 9

takes care of isotropic 8-dimensional quadratic spaces with trivial discriminant, while in Section 10 spaces of dimension 7 which represent their discriminant are examined. Section 11 considers general isotropic spaces of dimension 8, and Section 12 presents the results for fields in which the squares have index 1 or 2 in the group of non-zero elements in detail.

1 Finite Dimensional Algebras

Let \mathbb{F} be a field of characteristic different from 2. An *algebra* over \mathbb{F} is a ring R with identity together with an embedding of \mathbb{F} into the center of R (taking $1 \in \mathbb{F}$ to $1 \in R$). We shall only consider finite-dimensional algebras here, so that we shall write *algebra* for *finite-dimensional algebra* through. Wedderburn's Theorem states that any \mathbb{F} -algebra the product of simple rings R_i , and each such ring admits two maps, the *reduced norm* and the *reduced trace*, to its center, which is a finite field extension \mathbb{K} of \mathbb{F} . The latter map, denoted $Tr_{\mathbb{K}}^R$, is a homomorphism of additive groups. The former, which we denote $N_{\mathbb{K}}^R$, is multiplicative, yielding a group homomorphism from the group R^\times of invertible elements of R into \mathbb{K}^\times . The field extension \mathbb{K}/\mathbb{F} comes with its norm and trace maps $N_{\mathbb{F}}^{\mathbb{K}}$ and $Tr_{\mathbb{F}}^{\mathbb{K}}$, and the total norm and trace maps $N_{\mathbb{F}}^R$ and $Tr_{\mathbb{F}}^R$ are the compositions $N_{\mathbb{F}}^{\mathbb{K}} \circ N_{\mathbb{K}}^R$ and $Tr_{\mathbb{F}}^{\mathbb{K}} \circ Tr_{\mathbb{K}}^R$ respectively. The norm and trace from the total ring $\prod_i R_i$ into \mathbb{F} and the product of the $N_{\mathbb{F}}^{R_i}$ and the sum of the $Tr_{\mathbb{F}}^{R_i}$ respectively.

All the tensor products (of vector spaces or algebras) will be over \mathbb{F} , hence the index will be omitted. In case one of the multipliers in a tensor product is a commutative \mathbb{F} -algebra, we shall shorten using subscripts, such as $R_{\mathbb{E}}$ for $R \otimes \mathbb{E}$.

Many of our algebras will come with an involution, sending $x \in R$ to \bar{x} , which is \mathbb{F} -linear and inverts the order of products. Then R decomposes, as a vector space, as the direct sum of the space R^+ of *symmetric* elements which are invariant under the involution, and the space R^- of the *anti-symmetric* ones, which are inverted by it (see Section 2 of [KMRT], and recall that $ch\mathbb{F} \neq 2$). In fact, we shall encounter two cases. In the first case R will be simple with center \mathbb{F} (hence of dimension n^2 over \mathbb{F} for some number n called the *degree* of R), in which case the involution is said to be *of the first kind*. Then the involution can be either *orthogonal*, where R^+ has dimension $\frac{n(n+1)}{2}$ and R^- is of dimension $\frac{n(n-1)}{2}$, or *symplectic*, where the dimensions are interchanged.

The second case is where the center of R is a (étale) quadratic \mathbb{F} -algebra \mathbb{E} , which is either a field extension (which is separable by the assumption on $ch\mathbb{F}$) with Galois automorphism ρ (denoted $z \mapsto z^\rho$ for $z \in \mathbb{E}$), or $\mathbb{E} = \mathbb{F} \times \mathbb{F}$, with \mathbb{F} embedded diagonally and ρ interchanging the two coordinates. Note that for z in a quadratic \mathbb{F} -algebra \mathbb{E} we have $z^\rho = Tr_{\mathbb{F}}^{\mathbb{E}}(z) - z$. We shall encounter only such algebras which come from central simple algebras over \mathbb{F} , i.e., those R over \mathbb{E} such that there exists a central simple algebra S over \mathbb{F} , with involution $x \mapsto \bar{x}$, such that $R \cong S_{\mathbb{E}}$. Then we write y^ρ for the image of $y \in R \cong S_{\mathbb{E}}$ under $Id_S \otimes \rho$, and the involution in question, which is *of the second type* or *unitary*, is $y \mapsto \bar{y}^\rho$. In this case R has dimension $2n^2$ over \mathbb{F} , and the spaces R^+ and

R^- are both of dimension n^2 over \mathbb{F} (and are *not* vector spaces over \mathbb{E}). In case $\mathbb{E} = \mathbb{F} \times \mathbb{F}$ we have $R = S \times S$, and $\overline{(s, t)}^\rho = (\bar{t}, \bar{s})$.

For a finite-dimensional \mathbb{F} -algebra R , let $M_n(R)$ be the ring of $n \times n$ matrices over R . In case R is commutative, the reduced norm and trace from $M_n(R)$ into R are just the matrix determinant and matrix trace respectively. If furthermore $R = \mathbb{E}$ is a quadratic \mathbb{F} -algebra, we shorten $(M^t)^\rho = (M^\rho)^t$ to just $M^{t\rho}$, where M^t denotes the matrix which is the transpose of M . Similarly, for $z \in \mathbb{E}^\times$ we write $z^{-\rho}$ for $(z^{-1})^\rho = (z^\rho)^{-1}$.

A *quaternion algebra* B over \mathbb{F} is a central simple \mathbb{F} -algebra of degree 2. It comes with a natural (symplectic) involution, called the *main involution*, which we denote by $\iota : x \mapsto \bar{x} = \text{Tr}_{\mathbb{F}}^B(x) - x$ (or sometimes ι_B where the quaternion algebra will not be clear from the context). This is the only symplectic involution on B —see Proposition 2.21 of [KMRT]. Since $\text{ch}\mathbb{F} \neq 2$, every quaternion algebra is generated by two anti-commuting (traceless) elements with squares in \mathbb{F} . The algebra in which the squares of these elements are α and β respectively will be denoted $(\frac{\alpha, \beta}{\mathbb{F}})$. We may multiply each generator by an element of \mathbb{F}^\times , so that multiplying α or β by squares yields an isomorphic quaternion algebra. If \mathbb{E} is a quadratic \mathbb{F} -algebra and B is a quaternion algebra then the norms $N_{\mathbb{F}}^{\mathbb{E}}$ and $N_{\mathbb{F}}^B$ are quadratic functions, and we have

Lemma 1.1. *The equalities $N_{\mathbb{F}}^{\mathbb{E}}(z + w) = N_{\mathbb{F}}^{\mathbb{E}}(z) + N_{\mathbb{F}}^{\mathbb{E}}(w) + \text{Tr}_{\mathbb{F}}^{\mathbb{E}}(zw^\rho)$ and $N_{\mathbb{F}}^B(x + y) = N_{\mathbb{F}}^B(x) + N_{\mathbb{F}}^B(y) + \text{Tr}_{\mathbb{F}}^B(x\bar{y})$ hold for every z and w in \mathbb{E} and x and y in B .*

Proof. The equalities $N_{\mathbb{F}}^{\mathbb{E}}(t) = tt^\rho$ and $\text{Tr}_{\mathbb{F}}^{\mathbb{E}}(t) = t + t^\rho$ hold for every $t \in \mathbb{E}$, and we also have $N_{\mathbb{F}}^B(s) = s\bar{s}$ and $\text{Tr}_{\mathbb{F}}^B(s) = s + \bar{s}$ for every $s \in B$. The lemma follows directly from these equalities. \square

For such \mathbb{E} and B we denote \mathbb{E}_0 and B_0 the spaces of traceless elements in \mathbb{E} and B . These are the spaces \mathbb{E}^- and B^- with respect to the involutions ρ and ι respectively, and they have respective dimensions 1 and 3 (as ρ is unitary and ι is symplectic).

A quaternion algebra B over \mathbb{F} either *splits*, i.e., it is isomorphic to $M_2(\mathbb{F})$, or a division algebra. In the former case $\text{Tr}_{\mathbb{F}}^B$ is the matrix trace, $N_{\mathbb{F}}^B$ is the determinant, and ι_B is the adjoint involution $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$. A *splitting field* of B is an extension \mathbb{K} of \mathbb{F} such that the quaternion algebra $B_{\mathbb{K}}$ over \mathbb{K} splits. A quadratic extension \mathbb{K} of \mathbb{F} , with Galois automorphism σ , is a splitting field of B if and only if it can be embedded into B . By choosing an embedding, the subset of B which anti-commutes with \mathbb{K} (namely $x \in B$ such that $xz = z^\sigma x$ for all $z \in \mathbb{K}$) form a 1-dimensional subspace over \mathbb{K} , which is contained in B_0 . The choice of the square δ of an invertible element there (which is a representative of a class in $\mathbb{F}^\times / N_{\mathbb{F}}^{\mathbb{K}}(\mathbb{K}^\times)$) yields an embedding of B into $M_2(\mathbb{K})$ in which $z + wj \in B$ (with z and w from \mathbb{K}) is taken to $\begin{pmatrix} z & \delta w \\ w^\sigma & z^\sigma \end{pmatrix}$. We denote the image of this algebra as $(\mathbb{K}, \sigma, \delta)$. For this algebra we shall use

Lemma 1.2. *The action of $\sigma = \text{Id}_B \otimes \sigma$ on $B_{\mathbb{K}} = M_2(\mathbb{K})$ for $B = (\mathbb{K}, \sigma, \delta)$ is defined by $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto \begin{pmatrix} 0 & \delta \\ 1 & 0 \end{pmatrix} \begin{pmatrix} a^\sigma & b^\sigma \\ c^\sigma & d^\sigma \end{pmatrix} \begin{pmatrix} 0 & \delta \\ 1 & 0 \end{pmatrix} / \delta$.*

Proof. As $B_{\mathbb{K}} = M_2(\mathbb{K})$, it suffices to show that $(\mathbb{K}, \sigma, \delta)$ is the set of matrices which are stable under the operation which is asserted to be $Id_B \otimes \sigma$. As this operation takes $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ to $\begin{pmatrix} d^\sigma & \delta c^\sigma \\ b^\sigma/\delta & a^\sigma \end{pmatrix}$, we find that M is invariant under this operation if and only if $d = a^\sigma$ and $b = \delta c^\sigma$. As these conditions indeed characterize $(\mathbb{K}, \sigma, \delta)$, this proves the lemma. \square

For the split algebra $B = M_2(\mathbb{F})$ we have the following

Lemma 1.3. *For any $g \in M_2(\mathbb{F})$, conjugation by $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ takes \bar{g} and g^t to one another.*

Proof. When $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ we have $g^t = \begin{pmatrix} a & c \\ b & d \end{pmatrix}$ and $\bar{g} = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$. The result now follows from a simple calculation. \square

Lemma 1.3 will allow us to obtain equivalent models for our spin and Gspin groups, which are also used in the literature. Note that it relates a symplectic involution on $M_2(\mathbb{F})$ with an orthogonal one.

Another type of algebras which we shall encounter are *bi-quaternion algebras*, which are simple algebras A of degree 4 which may be presented as $A \cong B \otimes C$ where B and C are quaternion algebras. Given such a presentation, there is an involution $\iota_B \otimes \iota_C : x \mapsto \bar{x}$ on A , which is orthogonal. However, A may be presented as the tensor product of two quaternion algebras in many ways, each giving a different orthogonal involution. Moreover, not all the orthogonal involutions on A may be obtained in this way, and A admits also symplectic involutions—see Proposition 2.7 of [KMRT]. However, we shall use the notation \bar{x} for a bi-quaternion algebra only when the presentation as $B \otimes C$ is clear from the context.

We shall be needing subgroups of the groups of the form R^\times arising from simple \mathbb{F} -algebras R , which are defined by the norm. If \mathbb{K} is the center of R and $H \subseteq \mathbb{E}^\times$, we shall denote R^H the subgroup of \mathbb{E}^\times consisting of those elements $x \in R^\times$ such that $N_{\mathbb{E}}^R(x) \in H$. We shall extend this notation to algebras of the form $R = S \times S$ for a central simple \mathbb{F} -algebra S , with subgroups of $\mathbb{F}^\times \times \mathbb{F}^\times$. If \mathbb{E} is any commutative \mathbb{F} -algebra then \mathbb{E}^H is defined similarly. Note that for an algebra of the form $S_{\mathbb{E}}$ with S central simple over \mathbb{F} , we shall use the norm to the center \mathbb{E} and not to \mathbb{F} for defining $S_{\mathbb{E}}^H$.

The group of invertible matrices in $M_n(R)$ will be denoted $GL_n(R)$ also when R is not commutative. If R comes with an involution $x \mapsto \bar{x}$, then $M \mapsto \bar{M}^t$ is an involution on $M_n(R)$ of the same type as the involution on R . Note that if R is not commutative then $M \mapsto \bar{M}$ and $M \mapsto M^t$ do not behave well with respect to products. If R is simple then $M_n(R)$ is also simple, with the same center \mathbb{K} , and the group $M_n(R)^H$ for a subgroup $H \subseteq \mathbb{K}^\times$ will be denoted $GL_n^H(R)$. In case $H = \{1\}$ we shall use just the superscript 1 (with no brackets), and where $R = \mathbb{E}$ is a field extension of \mathbb{F} , we shall write $SL_n(\mathbb{E})$ for $GL_n^1(\mathbb{E})$.

2 Orthogonal and Other Groups

We shall be considering quadratic spaces over \mathbb{F} . As $ch\mathbb{F} \neq 2$, this is equivalent to spaces endowed with a symmetric bilinear form, so that we use the bilinear and quadratic forms interchangeably. All the spaces we consider will of (positive) finite dimension and non-degenerate, and these assumptions will be made even when not stated explicitly. Many of our vector spaces will be subsets of \mathbb{F} -algebras, so that we write the pairing (or product) of two vectors v and w of a quadratic space V as $\langle v, w \rangle$. Moreover, the number $\langle v, v \rangle$ will be written $|v|^2$ (in order to distinguish it from v^2 in the algebra involved), and will be called the *vector norm* (and not just norm) of v . In case confusion may arise as to which vector space is considered, we may write also $\langle v, w \rangle_V$ and $|v|_V^2$. We have

Lemma 2.1. *the equality $|v + w|^2 = |v|^2 + |w|^2 + 2\langle v, w \rangle$ holds for any v and w in V .*

Proof. The lemma follows directly from the definition of the vector norm and the symmetry of the bilinear form. \square

While no “absolute value” $|v|$ exists in general, we shall write the m th power of $|v|^2$ as $|v|^{2m}$. Two vectors v and w are said to be *orthogonal* or *perpendicular* if $\langle v, w \rangle = 0$, and v^\perp denotes the (1-codimensional) subspace of elements of V which are perpendicular to v . A vector $0 \neq v \in V$ is called *isotropic* if $|v|^2 = 0$, and *anisotropic* otherwise. A quadratic space V is called *isotropic* if it contains (non-zero) isotropic vectors, and *anisotropic* otherwise. An *orthogonal basis* is a basis consisting of vectors which are all orthogonal to one another (hence they must all be anisotropic), and every quadratic space admits such a basis since $ch\mathbb{F} \neq 2$. A *rescaling* of a quadratic space V is the same quadratic space but with all the pairings and vector norms multiplied by a global scalar from \mathbb{F}^\times . The *determinant* of a quadratic space V is defined as the determinant of a Gram matrix representing the bilinear form of V in some basis (which reduces to the product of the vector norms of an orthogonal basis) in $\mathbb{F}^\times/(\mathbb{F}^\times)^2$. However, it turns out more useful to consider the *discriminant* of V , which is the determinant multiplied by $(-1)^{n(n-1)/2}$ where n is the dimension of V . By some abuse of notation, we shall sometime treat the discriminant as an actual representative from \mathbb{F}^\times mapping to the appropriate class in $\mathbb{F}^\times/(\mathbb{F}^\times)^2$, but this will always be independent of the representative chosen. Note that the discriminant is invariant under rescaling if the dimension is even, but it is multiplied by the rescaling factor when the dimension is odd.

Given a quadratic space V , we define the *orthogonal group* $O(V)$ to be the group of linear transformations of V which preserves the bilinear form. Given an anisotropic vector $v \in V$, the map which inverts v and leaves the space v^\perp invariant belongs to $O(V)$. We call this map the *reflection inverting v* . The only property of $O(V)$ which we shall use here is the *Cartan–Dieudonné Theorem*, namely

Proposition 2.2. *The group $O(V)$ is generated by reflections.*

Proof. For a simple proof, see e.g., Corollary 4.3 of [MH]. \square

The determinant is a surjective homomorphism $O(V) \rightarrow \{\pm 1\}$ (as reflections have determinant -1) and the kernel, the *special orthogonal group*, is denoted $SO(V)$. It consists of those transformations which can be written as the product of an *even* number of reflections. There is a homomorphism $O(V) \rightarrow \mathbb{F}^\times / (\mathbb{F}^\times)^2$, called the *spinor norm*, which takes the reflection inverting an anisotropic vector v to the image of $|v|^2$ in $\mathbb{F}^\times / (\mathbb{F}^\times)^2$. As with the discriminant, we may sometimes say that an element of $O(V)$ has spinor norm $t \in \mathbb{F}^\times$, meaning that its spinor norm is $t(\mathbb{F}^\times)^2 \in \mathbb{F}^\times / (\mathbb{F}^\times)^2$. Note the rescaling the bilinear form leaves the spinor norms of elements of $SO(V)$ invariant, but multiplies those of elements in $O(V) \setminus SO(V)$ by the rescaling factor. Hence for $SO(V)$ the spinor norm is well-defined also when we consider V up to rescaling. The subgroup of $SO(V)$ consisting of elements having spinor norm 1 (i.e., a square) is denoted $SO^1(V)$. Note that the global inversion $-Id_V$ always has spinor norm which equals the discriminant of the space, in correspondence with $-Id_V$ being in $SO(V)$ (hence having a spinor norm which is invariant under rescalings) if and only if V has even dimension.

We shall define the *spin group* of V to be a double cover of $SO^1(V)$. The *Gspin group*, or the *even Clifford group*, of V is defined as a group mapping onto $SO(V)$ with kernel \mathbb{F}^\times . We wish to construct these groups, in low dimensions, without needing to investigate the Clifford algebra of V . This becomes much simpler after some normalization by rescaling. Therefore we do not consider groups like the pin group, $O^1(V)$, and the full Clifford group, which map onto subgroups of $O(V)$ which are not contained in $SO(V)$, as they are not invariant under rescaling.

Let \mathbb{E} be a quadratic extension of \mathbb{F} , with Galois automorphism ρ . A *unitary space* over \mathbb{E} (with respect to ρ) is a vector space (again finite-dimensional and non-trivial) vector space over \mathbb{E} with a (non-degenerate) Hermitian sesqui-linear form, where the conjugation is defined using ρ . Unitary spaces may be defined in terms Hermitian Gram matrices, which may always be reduced (by the choice of an appropriate basis) to regular diagonal matrices over the fixed field \mathbb{F} of ρ . A unitary space also has a determinant (and a discriminant), which are similarly defined and lie in $\mathbb{F}^\times / N_{\mathbb{F}}^{\mathbb{E}}(\mathbb{E}^\times)$. We shall allow ourselves the same abuse of notation for these unitary determinants and discriminants. We shall present unitary spaces only through representing Gram matrices (which are Hermitian and regular), and the *unitary group* of such a matrix M , denoted $U_{\mathbb{E},\rho}(M)$, is the group of linear transformations of a unitary spaces which preserve the sesqui-linear form defined by M , namely those $g \in GL_n(\mathbb{E})$ such that $gMg^{t\rho} = M$. Elements of $U_{\mathbb{E},\rho}(M)$ have determinants in \mathbb{E}^1 , and we define $SU_{\mathbb{E},\rho}(M)$ to be the subgroup of unitary matrices whose determinant is 1. Matrices which multiply the sesqui-linear form (hence M) by a scalar from \mathbb{E}^\times , which must then be in \mathbb{F}^\times , form the *general unitary group* $GU_{\mathbb{E},\rho}(M)$. Now, if $g \in GU_{\mathbb{E},\rho}(M)$ multiplies the sesqui-linear form (hence M) by a scalar $s = s(g) \in \mathbb{F}^\times$, and

the space has dimension n , then we have $N_{\mathbb{F}}^{\mathbb{E}}(\det g) = s(g)^n$. In case n is even, we define the group $GSU_{\mathbb{E},\rho}(M)$ consisting of elements g of the latter group such that $\det g = s(g)^{n/2}$ (note that this is not equivalent to the condition that $\det g \in \mathbb{F}^{\times}$ —see Lemma 7.2 and Corollary 7.8 below for an example with $n = 4$). It follows that $SU_{\mathbb{E},\rho}(M) = U_{\mathbb{E},\rho}(M) \cap GSU_{\mathbb{E},\rho}(M)$. All these groups are invariant under rescaling of M by an element of \mathbb{F}^{\times} . In small dimension we may relate the (general) unitary groups to other groups, as is seen in the following

Lemma 2.3. *If M is 1-dimensional then $GU_{\mathbb{E},\rho}(M)$, $U_{\mathbb{E},\rho}(M)$, and $SU_{\mathbb{E},\rho}(M)$ are \mathbb{E}^{\times} , \mathbb{E}^1 , and $\{1\}$ respectively. In the case of 2-dimensional, $GSU_{\mathbb{E},\rho}(M)$ is conjugate to B^{\times} for some quaternion algebra B over \mathbb{F} which is split over \mathbb{K} , and $SU_{\mathbb{E},\rho}(M)$ is conjugate to B^1 .*

Proof. In case M is just a scalar (which may be taken to be 1), the unitary relations on $z \in GL_1(\mathbb{E}) = \mathbb{E}^{\times}$ are just $zz^{\rho} \in \mathbb{F}^{\times}$ (which poses no further restriction on z), $zz^{\rho} = 1$, and $z = 1$ respectively. In the 2-dimensional case we may take M to be diagonal (this change might impose some conjugacy relation), and after rescaling we may assume that $M = \begin{pmatrix} -\varepsilon & 0 \\ 0 & 1 \end{pmatrix}$ where ε represents the discriminant of the unitary space. Now, multiplying the defining relation of $g \in GSU_{\mathbb{E},\rho}(M)$ by $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ from the right and using Lemma 1.3 transforms this relation to $g \begin{pmatrix} 0 & \varepsilon \\ 1 & 0 \end{pmatrix} \bar{g}^{\rho} = \det g \begin{pmatrix} 0 & \varepsilon \\ 1 & 0 \end{pmatrix}$. As $\det g = g\bar{g}$, the latter relation shows that \bar{g} is invariant under the relation from Lemma 1.2, implying that \bar{g} lies in $B = (\mathbb{E}, \rho, \varepsilon)$. As the latter algebra is closed under the adjoint involution (which restricts to its main involution), we find that $g \in B^{\times}$ as well. As $SU_{\mathbb{E},\rho}(M)$ is the subgroup of determinant 1 elements in $GSU_{\mathbb{E},\rho}(M)$, it is taken to B^1 in this map. This proves the lemma. \square

For a subgroup H of \mathbb{F}^{\times} , we write $GU_{\mathbb{E},\rho}^H(M)$, as well as $GSU_{\mathbb{E},\rho}^H(M)$ of n is even, for the subgroup of the appropriate groups consisting of matrices g whose multiplier $t(g)$ lies in H . Thus $GSU_{\mathbb{E},\rho}^H(M) = GSU_{\mathbb{E},\rho}(M) \cap GU_{\mathbb{E},\rho}^H(M)$. The same argument as in Lemma 2.3 shows that if M is 1-dimensional then $GU_{\mathbb{E},\rho}^H(M)$ is \mathbb{E}^H , while for 2-dimensional M the group $GSU_{\mathbb{E},\rho}^H(M)$ is conjugate to B^H for some quaternion algebra B over \mathbb{F} .

The (classical) *symplectic group* $Sp_{2n}(\mathbb{F})$ is the group consisting of those matrices $g \in GL_{2n}(\mathbb{F})$ such that $g \begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix} g^t = \begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix}$. More generally, a symplectic group is the group of linear transformations of a vector space (which must be of even dimension) which preserve a non-degenerate anti-symmetric bilinear form on it, but every such group is isomorphic (or conjugate) to the classical one. The *general symplectic group* $GSp_{2n}(\mathbb{F})$ consists of those matrices whose action multiplies $\begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix}$ by a scalar. If we consider a quadratic extension \mathbb{E} of \mathbb{F} with Galois automorphism ρ , then preserving an anti-Hermitian matrix via $g : M \mapsto gMg^{t\rho}$ is the same as preserving the Hermitian matrix which is obtained from M through multiplication by a scalar from \mathbb{E}_0 , so that no new groups are obtained in this way. On the other hand, using a central simple algebra R with an involution $x \mapsto \bar{x}$ of the first kind, one defines further

types of symplectic groups. We have the operation of $GL_n(R)$ on $M_n(R)$ via $g : M \mapsto gM\bar{g}^t$, and any matrix (Hermitian, anti-Hermitian, or neither) may be used to define such a group. Note that $g : M \mapsto gMg^t$ and $g : M \mapsto gM\bar{g}$ may not be used here, as they do not define actions of $M_n(R)$. Now, given any $M \in GL_n(R)$, we let $Sp_R(M)$ is the group of elements $g \in GL_n(R)$ which preserve M , and $GSp_R(M)$ consists of those matrices whose action multiplies M by a scalar from \mathbb{F}^\times . Note that rescaling M by a factor from \mathbb{F}^\times still does not change these groups. If B is a quaternion algebra (with its main involution) and M is Hermitian, we may choose a basis for our space such that M is diagonal, with entries from \mathbb{F} . In this case we just have

Lemma 2.4. *If M has dimension 1 then $GSp_B(M) = B^\times$ and $Sp_B(M) = B^1$.*

Proof. Indeed, M is just a scalar from \mathbb{F}^\times , which may be taken to be 1. Hence the GSp relation just states that $x\bar{x} \in \mathbb{F}^\times$ (which poses no restriction on the element $x \in GL_1(B) = B^\times$), and the Sp condition means $x\bar{x} = 1$. This proves the lemma. \square

In resemblance with the classical case, we shall use $Sp_{2n}(R)$ and $GSp_{2n}(R)$ for the case where M is the anti-Hermitian matrix $\begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix}$. As usual, for a subgroup $H \subseteq \mathbb{F}^\times$ we define $GSp_R^H(M)$ and $GSp_{2n}^H(R)$ to be the subgroup of $GSp_R(M)$ and $GSp_{2n}(R)$ in which the multiplier comes from H . The proof of Lemma 2.4 shows that if B is a quaternion algebra with its main involution and M is 1-dimensional and Hermitian then the first group is just B^H .

3 Dimension ≤ 3

In dimension 1 we have only one quadratic space (up to isomorphism), namely \mathbb{F} itself, and the bilinear form is determined by the norm of 1 (which is most naturally normalized to be 1). Proposition 2.2 shows that $O(\mathbb{F})$ is generated by the only reflection $-Id_{\mathbb{F}}$, so that it equals $\{\pm 1\}$ and $SO(\mathbb{F}) = \{1\}$. The spinor norm is just 1 on $SO(\mathbb{F})$ (i.e., $SO^1(\mathbb{F}) = SO(\mathbb{F})$). The $Gspin$ group is thus \mathbb{F}^\times , and the $spin$ group is $\{\pm 1\}$, both mapping to the trivial group $SO(\mathbb{F}) = \{1\}$.

For dimension 2 we define \mathbb{E} to be $\mathbb{F}(\sqrt{d})$, where d is the discriminant of the space. Our space is described by the following

Lemma 3.1. *Any 2-dimensional quadratic space of discriminant d is isometric to \mathbb{E} , with a rescaling of the quadratic form $N_{\mathbb{F}}^{\mathbb{E}}$. Rescaled appropriately, we get $2\langle z, w \rangle = Tr_{\mathbb{F}}^{\mathbb{E}}(zw^p)$ for z and w in \mathbb{E} .*

Proof. The second equality for \mathbb{E} with $|z|^2 = N_{\mathbb{F}}^{\mathbb{E}}(z)$ follows from Lemmas 1.1 and 2.1. Hence $1 \in \mathbb{E}$ satisfies $|1|^2 = 1$, an element from \mathbb{E}_0 has vector norm $-d$ (up to $(\mathbb{F}^\times)^2$), and they are orthogonal. Now, rescaling our original space space such that some anisotropic vector has vector norm 1, we find the orthogonal complement must be spanned by a vector whose vector norm is the determinant $-d$, just like in \mathbb{E} . This proves the lemma. \square

Multiplication from \mathbb{E}^1 preserves the vector norms, which defines a map $\mathbb{E}^1 \rightarrow O(\mathbb{E})$, which is clearly injective. However, in order to define the spin and Gspin group and be in the same spirit as the constructions for higher dimensions, we shall use

Lemma 3.2. *The action $g : z \mapsto gzg^{-\rho}$ defines a map $\mathbb{E}^\times \rightarrow O(\mathbb{E})$, with kernel \mathbb{F}^\times . The semi-direct product of $\text{Gal}_{\mathbb{F}}^{\mathbb{E}} = \{Id_{\mathbb{E}}, \rho\}$ with \mathbb{E}^\times also maps to $O(\mathbb{E})$.*

Proof. As $N_{\mathbb{F}}^{\mathbb{E}}(g^\rho) = N_{\mathbb{F}}^{\mathbb{E}}(g)$ and the norm is multiplicative, we have the equalities $|gzg^{-\rho}|^2 = |z|^2$ as well as $|z^\rho|^2 = |z|^2$. Hence both \mathbb{E}^\times and ρ map to $O(\mathbb{E})$. The kernel of the map from \mathbb{E}^\times consists of those elements of \mathbb{E}^\times such that $g = g^\rho$, which is \mathbb{F}^\times . The equality $(gzg^{-\rho})^\rho = g^\rho z^\rho g^{-1}$ shows that the map from the semi-direct product is also a homomorphism. This proves the lemma. \square

In fact, the map from \mathbb{E}^\times defined in Lemma 3.2 and the map defined above it have the same image, by Hilbert's Theorem 90. The next step is

Lemma 3.3. *Fix some $0 \neq h \in \mathbb{E}^0$. Then for every $g \in \mathbb{E}^\times$ the map taking $z \in \mathbb{E}$ to $(gh)z^\rho(gh)^{-\rho}$ is the reflection inverting g .*

Proof. As $h^\rho = -h$, this map takes z to $-gz^\rho g^{-\rho}$. It is clear that g is inverted (as $g^\rho = z^\rho$ cancels with $g^{-\rho}$). Now, elements which are perpendicular to g are those from $g\mathbb{E}_0$ (so that multiplying by g^ρ yields an element of $N_{\mathbb{F}}^{\mathbb{E}}(g)\mathbb{E}_0 = \mathbb{E}_0$, on which $\text{Tr}_{\mathbb{F}}^{\mathbb{E}}$ vanishes). They are all multiples of gh . As $(gh)^\rho = -hg^\rho$, a similar calculation shows that gh is invariant under this operation. This proves the lemma. \square

Using all this, we can now establish

Theorem 3.4. *A special orthogonal group of a 1-dimensional space is a one-element trivial group. For a 2-dimensional space of discriminant d , the Gspin group is \mathbb{E}^\times , and the spin and special orthogonal groups are isomorphic to \mathbb{E}^1 .*

Proof. The 1-dimensional part was already proven. Lemma 3.3 and Proposition 2.2 show that the map the semi-direct product defined in Lemma 3.2 surjects onto $O(\mathbb{E})$. As ρ represents an element of $O(\mathbb{E}) \setminus SO(\mathbb{E})$ (it inverts \mathbb{E}_0 and leaves \mathbb{F} invariant), Lemma 3.3 shows that \mathbb{E}^\times maps to $SO(\mathbb{E})$. As \mathbb{E}^\times has the same index 2 in the semi-direct product as $SO(\mathbb{E})$ has in $O(\mathbb{E})$, this map is also surjective, with kernel \mathbb{F}^\times . Hence \mathbb{E}^\times is $Gspin(\mathbb{E})$, and $SO(\mathbb{E})$ is isomorphic to \mathbb{E}^1 . Now, the fact that ρ has spinor norm $-d$ (as this is the vector norm of non-zero elements of \mathbb{E}_0 , up to $(\mathbb{F}^\times)^2$), implies that the image of $g \in \mathbb{E}^\times$ in $SO(\mathbb{E})$ has spinor norm $N_{\mathbb{F}}^{\mathbb{E}}(g)$: Indeed, Lemma 3.3 shows that its composition with ρ inverts an element of norm $-dN_{\mathbb{F}}^{\mathbb{E}}(g)$, and the spinor norm is a group homomorphism. Thus $SO^1(\mathbb{E})$ is the image of elements $g \in \mathbb{E}^{(\mathbb{F}^\times)^2}$, and as we may divide by elements of the kernel \mathbb{F}^\times of $\mathbb{E}^\times \rightarrow SO(\mathbb{E})$, it suffices to consider $g \in \mathbb{E}^1$. Hence the map $\mathbb{E}^1 \rightarrow SO^1(\mathbb{E})$, which is just $g \mapsto g^2$ since $g^{-\rho} = g$ for $g \in \mathbb{E}^1$, is surjective, and the kernel is just $\mathbb{E}^1 \cap \mathbb{F}^\times = \{\pm 1\}$. It follows that $spin(\mathbb{E}) = \mathbb{E}^1$ as well, and $SO^1(\mathbb{E})$ is the group $(\mathbb{E}^1)^2$ of squares of elements from \mathbb{E}^1 . This proves the proposition. \square

Another way to write the groups $Gspin(\mathbb{E})$ and $spin(\mathbb{E})$ are as $GU_{\mathbb{E},\rho}(1)$ and $U_{\mathbb{E},\rho}(1)$ respectively, by Lemma 2.3. We remark that when $SO(\mathbb{E})$ is given in terms of multiplication from \mathbb{E}^1 , the spinor norm of $u \in \mathbb{E}^1$, can be evaluated as $N_{\mathbb{F}}^{\mathbb{E}}(1+u)$ for $u \neq -1$ and d for $u = -1$ (note that the latter represents $-Id_{\mathbb{E}}$). To see this, write $u = \frac{g}{g^\rho}$ for $g \in \mathbb{E}^\times$, so that $1+u$ equals $\frac{Tr_{\mathbb{E}}^{\mathbb{E}}(g)}{N_{\mathbb{F}}^{\mathbb{E}}(g)}g$, and $N_{\mathbb{F}}^{\mathbb{E}}(1+u) \in N_{\mathbb{F}}^{\mathbb{E}}(g)(\mathbb{F}^\times)^2$ since $g \notin \mathbb{E}_0$ for $u \neq -1$. Comparing these models we find that an element $u \in \mathbb{E}^1$ (other than -1) lies in $(\mathbb{E}^1)^2$ if and only if $N_{\mathbb{F}}^{\mathbb{E}}(1+u) \in \mathbb{F}^\times$, a fact which may also be verified directly. The remaining element -1 lies in $SO^1(\mathbb{E})$ if and only if its spinor norm is a square, i.e., if $\mathbb{E} = \mathbb{F}(\sqrt{-1})$. It is easy to verify that $-1 \in (\mathbb{E}^1)^2$ precisely then this is indeed the case.

As a special case of Theorem 3.4 we obtain

Corollary 3.5. *A 2-dimensional quadratic space is isotropic if and only if it has a trivial discriminant. In this case the $Gspin$ group is to $\mathbb{F}^\times \times \mathbb{F}^\times$, while the spin and special orthogonal groups are isomorphic to \mathbb{F}^\times .*

Proof. By Lemma 3.1, an isotropic 2-dimensional quadratic space comes, up to rescaling, from a quadratic algebra contains non-zero norm 0 elements. But such an algebra cannot be a field, which is equivalent to d being a square. In this case $\mathbb{E} = \mathbb{F} \times \mathbb{F}$, so that $Gspin(\mathbb{F} \times \mathbb{F}) = \mathbb{E}^\times = \mathbb{F}^\times \times \mathbb{F}^\times$. The group \mathbb{E}^1 (which is the spin group) consists of the pairs $(r, \frac{1}{r})$ with $r \in \mathbb{F}^\times$, so it is isomorphic to \mathbb{F}^\times . As \mathbb{F}^\times is embedded in \mathbb{E}^\times diagonally, the quotient $SO(\mathbb{F} \times \mathbb{F})$ is the isomorphic image of the subgroup $\{(r, 1) | r \in \mathbb{F}^\times\}$, which is also a copy of \mathbb{F}^\times . This proves the corollary. \square

We remark that in the case presented in Corollary 3.5, the element $(r, \frac{1}{r})$ of \mathbb{E}^1 (considered as $SO(\mathbb{F} \times \mathbb{F})$ now) is $\frac{g}{g^\rho}$ for $g = (r, 1)$, so that its spinor norm is just r . This value coincides with the norm of $(1+r, 1+\frac{1}{r})$ for $r \neq -1$ and with $d = -1$ for $r = -1$. Hence $SO(\mathbb{F} \times \mathbb{F}) = \mathbb{F}^\times$ modulo $(\mathbb{F}^\times)^2$. The group $SO^1(\mathbb{F} \times \mathbb{F})$ is just $(\mathbb{F}^\times)^2$, given as the quotient of $Spin(\mathbb{F} \times \mathbb{F}) = \mathbb{E}^1 \cong \mathbb{F}^\times$ modulo $\{\pm 1\}$.

The space appearing in Corollary 3.5 is called a *hyperbolic plane*. It may also be generated by two isotropic vectors with non-zero pairing, so that it is isometric to all its rescalings. In fact, every isotropic quadratic space contains a hyperbolic plane, and the complement is uniquely determined up to isomorphism by the Witt Cancellation Theorem.

In dimension 3 we can assume, by rescaling, that our quadratic form has determinant in $(\mathbb{F}^\times)^2$ (hence discriminant -1). Such a vector space (a 3-dimensional quadratic space with discriminant 1) will be called a *traceless quaternionic space* over \mathbb{F} , for a reason to be explained by the following

Lemma 3.6. *If B is a quaternion algebra over \mathbb{F} , then the space B_0 with the vector norm $|x|^2 = N_{\mathbb{F}}^B(x)$ is a traceless quaternionic space. Every traceless quaternionic space is isometric to a space which obtained in this way from some quaternion algebra B . The pairing on such a space is given by $2\langle x, y \rangle = Tr_{\mathbb{F}}^B(x\bar{y})$ for x and y in B_0 .*

Proof. The latter formula for the pairing on B_0 is a consequence of Lemmas 1.1 and 2.1 (in fact, the same formula holds for x and y in B with this quadratic form). Now, two elements of B_0 are orthogonal if and only if they anti-commute. Writing B is $(\frac{\alpha, \beta}{\mathbb{F}})$, we get two such elements having norms $-\alpha$ and $-\beta$, and as their product is orthogonal to both of them of squares to $-\alpha\beta$ (hence with norm $+\alpha\beta$), we get the required determinant in $(\mathbb{F}^\times)^2$. Conversely, if two orthogonal elements of a traceless quaternionic space have norms $-\alpha$ and $-\beta$ respectively, then the determinant condition shows that a generator for their orthogonal complement can be normalized to have norm $+\alpha\beta$, and this space is isometric to B_0 for $(\frac{\alpha, \beta}{\mathbb{F}})$. This proves the lemma. \square

Regarding the group acting here we have

Lemma 3.7. *The group B^\times is mapped into $O(B_0)$ via $g : u \mapsto gu\bar{g}/N_{\mathbb{F}}^B(g)$, with kernel \mathbb{F}^\times . Letting -1 operate as $-Id_{B_0}$ yields a map from the direct product $B^\times \times \{\pm 1\}$ into $O(B_0)$.*

Proof. Since $N_{\mathbb{F}}^B(g) = \bar{g}g$, g maps u to gug^{-1} , and the multiplicativity of the norm shows that $|gug^{-1}|^2 = |u|^2$. An element lies in the kernel if and only if it is central (since with the complement \mathbb{F} of B_0 in B it does commute), so that the kernel is indeed \mathbb{F}^\times . The centrality of $-Id_{B_0}$ in $O(B_0)$ yields the last assertion. This proves the lemma. \square

We remark that the operation of -1 coincides with the operation of the main involution of B . the analysis of the orthogonal group begins with the following

Lemma 3.8. *If $g \in B_0$ has non-zero vector norm, then the orthogonal transformation taking $u \in B_0$ to $-\frac{gu\bar{g}}{N_{\mathbb{F}}^B(g)}$ is the reflection inverting g .*

Proof. The proof of Lemma 3.6 shows that conjugation by g inverts g^\perp , and it clearly leaves g invariant. Composing with the central map $-Id_{B_0}$, we establish the lemma. \square

We can now prove

Theorem 3.9. *The Gspin group of a 3-dimensional \mathbb{F} -space is the group B^\times for some quaternion algebra B over \mathbb{F} , which is generated by invertible elements of B_0 . The spin is the subgroup B^1 arising from this quaternion algebra B .*

Proof. Proposition 2.2 and Lemma 3.8, the operation in which is the action, from Lemma 3.7, of g on $-u$, imply that the map $B^\times \times \{\pm 1\} \rightarrow O(B_0)$ is surjective. As reflections and $-Id_{B_0}$ has determinant -1 , the image of B^\times lies in $SO(B_0)$, and index considerations show that the map $B^\times \rightarrow SO(B_0)$ is surjective. As the kernel is \mathbb{F}^\times , we find that $Gspin(B_0) = B^\times$. Taking out the action of the central element $-Id_{B_0}$, Lemma 3.8 shows that $B_0 \cap B^\times$ indeed generates B^\times (a fact which in this case is easily verified directly), since the full kernel \mathbb{F}^\times is clearly generated by this set: $t = (tg)g^{-1}$ for $t \in \mathbb{F}^\times$. As for spinor norms, we first observe that under our normalization $-Id$ has spinor norm 1. Hence Lemma 3.8 and the fact that $|g|^2 = N_{\mathbb{F}}^B(g)$ imply that the spinor norm

of any $g \in B_0 \cap B^\times$ is $N_{\mathbb{F}}^B(g)$. Since such elements were seen to generate B^\times , the spinor norm of any $g \in B^\times$ is $N_{\mathbb{F}}^B(g)$. The group $SO^1(B_0)$ is thus the image of elements having reduced norms in $(\mathbb{F}^\times)^2$, and by appropriate scalar multiplication we may restrict to elements from B^1 . As the only scalars in B^1 are ± 1 , we find that B^1 is indeed $spin(B_0)$. This proves the proposition. \square

Lemmas 2.3 and 2.4 shows that the Gspin group from Theorem 3.9 can also be described as $GSp_B(1)$ and as $GSU_{\mathbb{K},\sigma}(\begin{smallmatrix} -\varepsilon & 0 \\ 0 & 1 \end{smallmatrix})$, in case $\mathbb{K} = \mathbb{F}(\eta)$ is a quadratic extension of \mathbb{F} (with Galois automorphism σ) which splits B , and $\varepsilon \in \mathbb{F}$ is such that $B \cong (\frac{\eta, \varepsilon}{\mathbb{F}})$. They also imply that the spin group in question is isomorphic to $Sp_B(1)$, as well as to $SU_{\mathbb{K},\sigma}(\begin{smallmatrix} -\varepsilon & 0 \\ 0 & 1 \end{smallmatrix})$ for such \mathbb{K} , σ , and ε .

The isotropic case in dimension 3 is given in

Corollary 3.10. *A quadratic space of dimension 3 is isotropic if and only if it is related to the split quaternion algebra $B = M_2(\mathbb{F})$. The Gspin group $Gspin(M_2(\mathbb{F})_0)$ is then $GL_2(\mathbb{F})$, and $spin(M_2(\mathbb{F})_0) = SL_2(\mathbb{F})$. We also have $SO(M_2(\mathbb{F})_0) = PGL_2(\mathbb{F})$ and $SO^1(M_2(\mathbb{F})_0) = PSL_2(\mathbb{F})$.*

Proof. If B_0 is isotropic then B cannot be a division algebra, and the space $M_2(\mathbb{F})_0$ does split. The Gspin and spin groups are determined by Theorem 3.9 with $B = M_2(\mathbb{F})$, and dividing the former by \mathbb{F}^\times and the latter by $\{\pm 1\}$ yields the asserted projective groups. This proves the corollary. \square

Lemma 2.3 allows us to write the Gspin and spin groups from Corollary 3.10 also as $GSU_{\mathbb{K},\sigma}(\begin{smallmatrix} -1 & 0 \\ 0 & 1 \end{smallmatrix})$ and $SU_{\mathbb{K},\sigma}(\begin{smallmatrix} -1 & 0 \\ 0 & 1 \end{smallmatrix})$ respectively, for any quadratic extension \mathbb{K} of \mathbb{F} (with Galois automorphism σ), since every such \mathbb{K} splits $M_2(\mathbb{F})$. In this case matrix transposition is also an element of $O(M_2(\mathbb{F})_0) \setminus SO(M_2(\mathbb{F})_0)$, and this element arises as the composition of $-Id$ and conjugation by $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ (see Lemma 1.3).

In this split case there is an additional assertion, which is given by

Corollary 3.11. *The groups $GL_2(\mathbb{F})$, $SL_2(\mathbb{F})$, $PGL_2(\mathbb{F})$, and $PSL_2(\mathbb{F})$ are the Gspin, spin, special orthogonal, and spinor norm kernel groups of the space $M_2^{sym}(\mathbb{F})$ of symmetric 2×2 matrices over \mathbb{F} , on which they all operate via $g : X \mapsto \frac{gXg^t}{\det g}$.*

Proof. As right multiplication by $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ takes $M_2(\mathbb{F})_0$ to $M_2^{sym}(\mathbb{F})$ and preserves determinants, the corollary follows from Corollary 3.10 and Lemma 1.3. \square

Note that the operation from Corollary 3.11 replaces the symplectic main involution on $B = M_2(\mathbb{F})$ by an orthogonal one, and the usual space B^- by an $M_2(\mathbb{F})^+$ space. The possible generators of $O(M_2^{sym}(\mathbb{F}))/SO(M_2^{sym}(\mathbb{F}))$ are again $-Id$ (which is now $X \mapsto -X^t$) and the adjoint involution.

4 Dimension 4

Given a quadratic space of dimension 4 and discriminant d over \mathbb{F} , we define \mathbb{E} to be $\mathbb{F}(\sqrt{d})$, with automorphism ρ . Given a quaternion algebra B over \mathbb{F} , the tensor product $B_{\mathbb{E}}$ comes endowed with the (unitary) involution $\iota \otimes \rho : x \mapsto \bar{x}^{\rho}$. Our space is given by

Lemma 4.1. *The space $B_{\mathbb{E}}^{-}$ of this involution becomes, when endowed with the quadratic form $|x|^2 = N_{\mathbb{E}}^{B_{\mathbb{E}}}(x)$, a quadratic space over \mathbb{F} with discriminant d , in which $2\langle x, y \rangle = \text{Tr}_{\mathbb{E}}^{B_{\mathbb{E}}}(x\bar{y})$ holds for every x and y . Every such space is obtained, up to rescaling in this way.*

Proof. $B_{\mathbb{E}}^{-}$ is contained in the quadratic space $B_{\mathbb{E}}$ over \mathbb{E} with the same vector norm, and the combination of Lemmas 1.1 and 2.1 shows that the formula for the pairing holds (in \mathbb{E}) for any two elements of the larger space. Now, $B_{\mathbb{E}}^{-}$ is $\mathbb{E}_0 \oplus B_0$ inside $B_{\mathbb{E}}$, and the direct sum is thus orthogonal inside there. Since $N_{\mathbb{E}}^{B_{\mathbb{E}}}$ coincides with $N_{\mathbb{F}}^B$ on B_0 and is the square map on \mathbb{E}_0 (both \mathbb{F} -valued), we find that $|x|^2 \in \mathbb{F}$ for every $x \in B_{\mathbb{E}}^{-}$. Moreover, B_0 has determinant 1 and \mathbb{E}_0 is spanned by an element h with $|h|^2 = h^2 = d$, so that the determinant and discriminant of such a space are d . Conversely, given a quadratic space of discriminant d , we may rescale our it such that a anisotropic element v of our choice has vector norm d . The subspace v^{\perp} is a traceless quaternionic space, so that by Lemma 3.6 it can be presented as B_0 for a quaternion algebra B over \mathbb{F} . Hence we found a presentation of our space as $\mathbb{E}_0 \oplus B_0$, namely $B_{\mathbb{E}}^{-}$, with $|x|^2 = N_{\mathbb{E}}^{B_{\mathbb{E}}}(x)$. This proves the lemma. \square

Note that the proof of Lemma 4.1 involved a choice of a vector, and choosing another vector (with the appropriate rescaling) may lead to other quaternion algebras which are not isomorphic to B . However, $B_{\mathbb{E}}$ remains the same algebra, but with a different unitary involution. The correspondence between quaternion algebras over \mathbb{F} which are contained in $B_{\mathbb{E}}$ and generate it over \mathbb{E} and unitary involutions on $B_{\mathbb{E}}$ is given in Proposition 2.22 of [KMRT]. However, we shall consider, B , $B_{\mathbb{E}}$, and the involution as fixed. As ι is canonical, this gives an interpretation of ρ on $B_{\mathbb{E}}$ as well. We also remark that the complementary space $B_{\mathbb{E}}^{+}$ is obtained from $B_{\mathbb{E}}^{-}$ via multiplication by an element of \mathbb{E}_0 , so that it is isometric to a rescaling of $B_{\mathbb{E}}^{-}$ as a quadratic space over \mathbb{F} .

$B_{\mathbb{E}}$ operates on itself by $g : x \mapsto gx\bar{g}^{\rho}$, and this action preserves the eigenspaces $B_{\mathbb{E}}^{\pm}$. Moreover, the two spaces are invariant under ι . Using this, we now prove

Lemma 4.2. *The group $B_{\mathbb{E}}^{\mathbb{F}^{\times}}$ maps into $O(B_{\mathbb{E}}^{-})$ if an element g operates as $x \mapsto \frac{gx\bar{g}^{\rho}}{N_{\mathbb{E}}^{B_{\mathbb{E}}}(g)}$. The kernel of this map is \mathbb{F}^{\times} . Let $\tilde{\iota}$ be the non-trivial element of a cyclic group of order 2, which operates on $B_{\mathbb{E}}^{\mathbb{F}^{\times}}$ as ρ . Sending $\tilde{\iota}$ to operate as ι defines a group homomorphism from the semi-direct product of $\{1, \tilde{\iota}\}$ and $B_{\mathbb{E}}^{\mathbb{F}^{\times}}$ to $O(B_{\mathbb{E}}^{-})$.*

Proof. For $g \in B_{\mathbb{E}}^{\mathbb{F}^\times}$ we may replace $\frac{\bar{g}^\rho}{N_{\mathbb{E}}^{B_{\mathbb{E}}}(g)}$ by just $g^{-\rho}$, and the fact that $N_{\mathbb{E}}^{B_{\mathbb{E}}}$ is multiplicative implies that the equality $\left| \frac{gx\bar{g}^\rho}{N_{\mathbb{E}}^{B_{\mathbb{E}}}(g)} \right|^2 = |x|^2$ holds for every $x \in B_{\mathbb{E}}^-$. An element g is in the kernel of this action if and only if it is central and satisfies $gg^\rho = N_{\mathbb{E}}^{B_{\mathbb{E}}}(g) = g^2$, which is equivalent to g being in \mathbb{F}^\times . We also know that ι preserves $B_{\mathbb{E}}^-$ and $|\bar{x}|^2 = |x|^2$ for elements of that space. The equality $gx\bar{g}^\rho = g^\rho x\bar{g}$ shows that the map to $O(B_{\mathbb{E}}^-)$ respects the product rule of the semi-direct product, which completes the proof of the lemma. \square

We remark that other elements of $B_{\mathbb{E}}^\times$ do not increase the image of the map $B_{\mathbb{E}}^{\mathbb{F}^\times} \rightarrow O(B_{\mathbb{E}}^-)$ from Lemma 4.2. Indeed, if $g \in B_{\mathbb{E}}^\times$ and $t \in \mathbb{E}^\times$ are such that $x \mapsto \frac{gx\bar{g}^\rho}{t}$ preserves $B_{\mathbb{E}}^-$ and is orthogonal, then $t \in \mathbb{F}^\times$, the number $\frac{N_{\mathbb{E}}^{B_{\mathbb{E}}}(g)}{t}$ lies in \mathbb{E}^1 hence equals $\frac{s^\rho}{s}$ for some $s \in \mathbb{E}^\times$ by Hilbert's Theorem 90. But then $sg \in B_{\mathbb{E}}^{\mathbb{F}^\times}$ and operates like $x \mapsto \frac{gx\bar{g}^\rho}{t}$. This assertion will also follow from Theorem 4.4 below. As for the kernel, note that a non-zero element $r \in \mathbb{E}_0$ is also central and lies $B_{\mathbb{E}}^{\mathbb{F}^\times}$, but as $rr^\rho = -r^2$ and $N_{\mathbb{E}}^{B_{\mathbb{E}}}(r) = r^2$, such elements operate as $-Id$ rather than trivially.

The properties of the larger homomorphism from Lemma 4.2 will follow from

Lemma 4.3. *For an element $g \in B_{\mathbb{E}}^- \cap B_{\mathbb{E}}^\times$, the reflection inverting g lies in the image of the map from Lemma 4.2, being $x \mapsto \frac{gx\bar{g}^\rho}{N_{\mathbb{E}}^{B_{\mathbb{E}}}(g)}$.*

Proof. The proof of Lemma 4.1 shows that every such g is in $B_{\mathbb{E}}^{\mathbb{F}^\times}$, so that the latter transformation comes from the semi-direct product appearing in Lemma 4.2. Now, $\bar{g}^\rho = -g$ for $g \in B_{\mathbb{E}}^-$, and $N_{\mathbb{E}}^{B_{\mathbb{E}}}(g) = \bar{g}g$. Thus, for $x = g$ the result of the action is $-g$, while if $x \in g^\perp$ the equality from Lemma 4.2 allows us to replace $-g\bar{x}$ by $+x\bar{g}$, so that the total expression is just x . This proves the lemma. \square

The groups obtained in dimension 4 are now given in the following

Theorem 4.4. *The quadratic space $B_{\mathbb{E}}^-$ has Gspin group $B_{\mathbb{E}}^{\mathbb{F}^\times}$, and it is generated by $B_{\mathbb{E}}^- \cap B_{\mathbb{E}}^\times$. The spin group is $B_{\mathbb{E}}^1$.*

Proof. The surjectivity of the map from the semi-direct product from Lemma 4.2 onto $O(B_{\mathbb{E}}^-)$ follows from Lemma 4.3, and Proposition 2.2. The fact that ι has determinant -1 in $O(B_{\mathbb{E}}^-)$ and index considerations show that $B_{\mathbb{E}}^{\mathbb{F}^\times}$ maps onto $SO(B_{\mathbb{E}}^-)$. The kernel of latter map being \mathbb{F}^\times by Lemma 4.2, we find that $Gspin(B_{\mathbb{E}}^-) = B_{\mathbb{E}}^{\mathbb{F}^\times}$. The semi-direct product structure shows, with Lemma 4.3, that $B_{\mathbb{E}}^- \cap B_{\mathbb{E}}^\times$ generates $B_{\mathbb{E}}^{\mathbb{F}^\times}$ (since again the kernel \mathbb{F}^\times is not a problem), a fact which also in this case may still be verified directly. Now, ι reflects the traceless quaternionic space B_0 , of determinant 1, so that its spinor norm is 1. Lemma 4.3 thus implies that the spinor norm of $g \in B_{\mathbb{E}}^- \cap B_{\mathbb{E}}^\times$ is $|g|^2 = N_{\mathbb{E}}^{B_{\mathbb{E}}}(g)$. As such elements were seen to generate $B_{\mathbb{E}}^{\mathbb{F}^\times}$, the multiplicativity of the norm implies that the spinor norm of any $g \in B_{\mathbb{E}}^{\mathbb{F}^\times}$ is $N_{\mathbb{E}}^{B_{\mathbb{E}}}(g)$. Elements whose (spinor, hence

algebra) norms lie in $(\mathbb{F}^\times)^2$ can be dividing by scalars from the kernel \mathbb{F}^\times , and land in $B_{\mathbb{E}}^1$. Thus $B_{\mathbb{E}}^1$ maps surjectively onto $SO^1(B_{\mathbb{E}}^-)$, and the kernel consists of those scalars whose norm (hence square) is 1. As these are just ± 1 , $B_{\mathbb{E}}^1$ is the spin group, which completes the proof of the proposition. \square

In addition to ι , ρ also represents an element of $O(B_{\mathbb{E}}^-)$. Its spinor norm is d , being the composition of ι and $-Id$ as well as being the reflection in a generator of \mathbb{E}_0 . However, ι is a more canonical representative of $O(B_{\mathbb{E}}^-)/SO(B_{\mathbb{E}}^-)$, being independent of the \mathbb{F} -structure on $B_{\mathbb{E}}$. Note that Theorems 3.9 and 4.4 imply that if d is not a square then any spin group arising from a 4-dimensional quadratic space over \mathbb{F} with discriminant d is isomorphic to the spin group of a suitable 3-dimensional quadratic space over $\mathbb{E} = \mathbb{F}(\sqrt{d})$. The converse does not hold though, as we need the quaternion algebra over \mathbb{E} to come from one over \mathbb{F} .

Recall now that every quaternion algebra B over \mathbb{F} becomes, with $|x|^2 = N_{\mathbb{F}}^B$, a quadratic space of discriminant 1. Lemma 4.1 and Theorem 4.4 has the following

Corollary 4.5. *Any 4-dimensional space of discriminant 1 is isometric to some quaternion algebra B over \mathbb{F} with its reduced norm (perhaps rescaled). The $Gspin$ group $Gspin(B)$ consists of those pairs in $B^\times \times B^\times$ having the same reduced norm, operating via left multiplication and inverted right multiplication, and $spin(B) = B^1 \times B^1$ with the same operation.*

Proof. The condition $d \in (\mathbb{F}^\times)^2$ implies that $\mathbb{E} = \mathbb{F} \times \mathbb{F}$. Hence $B_{\mathbb{E}} = B \times B$, and $B_{\mathbb{E}}^-$ is the subspace $\{(x, -\bar{x}) | x \in B\}$ of $B \times B$. As $N_{\mathbb{E}}^{B_{\mathbb{E}}}(x, -\bar{x}) = N_{\mathbb{F}}^B(x)$, the first assertion follows from Lemma 4.1. Now, $B_{\mathbb{E}}^{\times}$ consists, by definition, of those pairs $(g, h) \in B^\times \times B^\times$ such that $N_{\mathbb{E}}^B(g) = N_{\mathbb{F}}^B(h)$, sending $(x, -\bar{x})$ to $(gx\bar{h}, -h\bar{x}\bar{g})$ (the second entry being also $-gx\bar{h}$), divided by the common norm of g and h . In terms of the operation with $(g, h)^{-\rho}$ from the proof of Lemma 4.1, the action on B is given by $(g, h) : x \mapsto gxh^{-1}$. Restricting to elements of norm 1, the spin group is seen to be $B^1 \times B^1$. This proves the corollary. \square

The spin group from Corollary 4.5 is the product of two copies of a spin group of a 3-dimensional space over \mathbb{F} , which complements the relation to spaces over the quadratic extension \mathbb{E} in the other discriminants. Lemmas 2.3 and 2.4 allow us to write such spin groups as $Sp_B(1) \times Sp_B(1)$, as well as $SU_{\mathbb{K}, \sigma}(\begin{smallmatrix} -\varepsilon & 0 \\ 0 & 1 \end{smallmatrix}) \times SU_{\mathbb{K}, \sigma}(\begin{smallmatrix} -\varepsilon & 0 \\ 0 & 1 \end{smallmatrix})$ in case B is isomorphic to $(\frac{\eta, \varepsilon}{\mathbb{F}})$ and $\mathbb{K} = \mathbb{F}(\eta)$ with Galois automorphism σ .

We note that the \mathbb{F} -structure on $B_{\mathbb{E}}$, i.e., the quaternion algebra over \mathbb{F} which yields $B_{\mathbb{E}}$ after tensoring with \mathbb{E} , is equivalent to the choice of the automorphism on $B_{\mathbb{E}}$ which we denoted here also by ρ , or equivalently, by the involution $x \mapsto \bar{x}^\rho$ on $B_{\mathbb{E}}$ (see Proposition 2.22 of [KMRT]). Proposition 2.18 shows that all these involutions are related, and are in one-to-one correspondence with the space $(B_{\mathbb{E}}^+ \cap B_{\mathbb{E}}^\times)/\mathbb{F}^\times$, since every such involution is $x \mapsto b\bar{x}^\rho b^{-1}$ for invertible $b \in B_{\mathbb{E}}^+$ which is uniquely determined up to multiplication from \mathbb{F}^\times . However, we shall not need these results in what follows.

As for isotropy, here we have

Corollary 4.6. *The space $B_{\mathbb{E}}^-$ is isotropic if and only if \mathbb{E} splits B . We may then take $B = M_2(\mathbb{F})$. $Gspin(M_2(\mathbb{E})^-)$ is $GL_2^{\mathbb{F}^\times}(\mathbb{E})$, the spin group $spin(M_2(\mathbb{E})^-)$ as $SL_2(\mathbb{E})$, and $SO^1(M_2(\mathbb{E})^-)$ as $PSL_2(\mathbb{E})$.*

Proof. If $B_{\mathbb{E}}^-$ is isotropic then $B_{\mathbb{E}}$ cannot be a division algebra. Conversely, if \mathbb{E} splits B then there is an embedding $i : \mathbb{E} \rightarrow B$, and if $r \in \mathbb{E}_0$ then $r+i(r)$ belongs to $B_{\mathbb{E}}^-$ and is a zero-divisor (hence isotropic). By splitting a hyperbolic plane and rescaling a vector v which is perpendicular to this hyperbolic plane to have vector norm d . Then v^\perp is isotropic, and the corresponding quaternion algebra B splits by Corollary 3.10. The $Gspin$ and spin groups are given in Theorem 4.4 (written in terms of a split algebra), and the assertion about $SO^1(M_2(\mathbb{E})^-)$, which is $spin(M_2(\mathbb{E})^-)/\{\pm 1\}$, is immediate. This proves the corollary. \square

Also here we can consider matrix conjugation an element of $O(M_2(\mathbb{E})^-)$ of determinant 1, and it comes as the composition of the main involution (adjoint) and conjugation by $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ by Lemma 1.3. In this case we can get an equivalent quadratic space, as is given in the following

Corollary 4.7. *We can consider the groups $GL_2^{\mathbb{F}^\times}(\mathbb{E})$ and $SL_2(\mathbb{E})$ as the $Gspin$ and spin groups of the space $M_2^{Her}(\mathbb{E}, \rho)$ of 2×2 matrices of \mathbb{E} which are Hermitian with respect to ρ , with the vector norm being the determinant. The operation is via $g : X \mapsto \frac{gXg^{t\rho}}{\det g}$. In the case of trivial discriminant, we may consider the subgroup of $GL_2(\mathbb{F}) \times GL_2(\mathbb{F})$ consisting of pairs of matrices of the same determinant and $SL_2(\mathbb{F}) \times SL_2(\mathbb{F})$ as the $Gspin$ and the spin groups of $M_2(\mathbb{F})$ with the determinant also via the action $(g, h) : M \mapsto gMh^t$ divided by the common norm of g and h (which is trivial on the latter group).*

Proof. The first assertion follows directly from Corollary 3.10 and Lemma 1.3, since right multiplication by $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ takes $M_2(\mathbb{E})^-$ to $M_2^{Her}(\mathbb{E})$. The second assertion is obtained from the same considerations together with Corollary 4.5. \square

The quotient group $O(M_2^{Her}(\mathbb{E}, \rho))/SO(M_2^{Her}(\mathbb{E}, \rho))$, as well as the group $O(M_2(\mathbb{F}))/SO(M_2(\mathbb{F}))$, is again generated by adjoint or transposition, but here ρ coincides with the latter transformation.

5 Dimension 6, Discriminant 1

Our presentation of 6-dimensional spaces of discriminant 1 is based on presentations of bi-quaternion algebras over \mathbb{F} as tensor products of two quaternion algebras, as in

Lemma 5.1. *For two quaternion algebras over \mathbb{F} , B and C say, the subspace $(B_0 \otimes 1) \oplus (1 \otimes C_0)$ of the bi-quaternion algebra $A = B \otimes C$ is a quadratic space of dimension 6 and discriminant 1 if we define $|x \otimes 1 + 1 \otimes y|^2 = N_{\mathbb{F}}^B(x) - N_{\mathbb{F}}^C(y)$.*

Every 6-dimensional quadratic space of discriminant 1 may be obtained, up to isometries and rescalings, in this way.

Proof. Lemma 3.6 shows that $(B_0 \otimes 1) \oplus (1 \otimes C_0)$ is the direct sum of two 3-dimensional spaces of determinants 1 and -1 , so that the total determinant is -1 and the discriminant is 1. Conversely, given a quadratic space V of dimension 6 and discriminant 1, choose a 3-dimensional non-degenerate subspace of V , and rescale V such that the chosen space has determinant 1. The discriminant 1 condition implies that the orthogonal complement is also a traceless quaternionic space but rescaled by -1 , so that the assertion follows from Lemma 3.6. This proves the lemma. \square

The space from Lemma 5.1 is called the *Albert form* of the two quaternion algebras B and C . The bi-quaternion algebra $A = B \otimes C$ comes with the involution $\iota_B \otimes \iota_C$, which depends on the presentation of A as $B \otimes C$. When we decompose A as $A^+ \oplus A^-$ according to this involution, then A^+ is the 10-dimensional space $B_0 \otimes C_0 \oplus \mathbb{F}(1 \otimes 1)$, and A^- is the space $(B_0 \otimes 1) \oplus (1 \otimes C_0)$ from Lemma 5.1. In particular, $\iota_B \otimes \iota_C$ is orthogonal—compare Proposition 2.23(1) of [KMRT]. Note that the product structure on our original space depends on the choice of the 3-dimensional space which we normalize to be $B_0 \otimes 1$: Observe that $B_0 \otimes 1$ and $1 \otimes C_0$ are precisely those elements of A^- whose algebra square lies in \mathbb{F} .

Several remarks are in order here. First, there are many involutions, orthogonal and symplectic, on A , and to each of them one may associate a quadratic 6-dimensional space of discriminant 1 (an Albert form). Two Albert forms are isometric up to rescaling if and only if they come from isomorphic bi-quaternion algebras, by a result of [J] (Lemma 5.1 already shows that this construction gives all the Albert form, up to rescaling). However, not all the involutions on A , and not even all the orthogonal involutions on A , come from a presentation of A as $B \otimes C$, though the mere existence of an involution of the first kind on a degree 4 central simple algebra A implies that A is the tensor product of two quaternion algebras over \mathbb{F} (a theorem of Albert—see Theorem 16.1 of [KMRT]). However, we stick to one fixed involution, which does arise in this way, and our results are independent of all the results mentioned in this paragraph. We also remark that Section 16 of [KMRT] presents results which are very similar to ours, but some of the calculations do not appear there (in particular, the main calculation required for Proposition 5.5 below), and we concentrate on a case where many technical aspects become simpler.

Our analysis is based on the following

Lemma 5.2. *If $\theta : A^- \rightarrow A^-$ takes $u = x \otimes 1 + 1 \otimes y$ to $\tilde{u} = -x \otimes 1 + 1 \otimes y$ then $u\tilde{u} = \tilde{u}u = |u|^2$ in A , and $2\langle v, w \rangle = v\tilde{w} + w\tilde{v} = \tilde{v}w + \tilde{w}v$.*

Proof. The first assertion follows from a simple and direct calculation. The second equality then follows from Lemma 2.1. This proves the lemma. \square

Note that θ might be considered as the restriction of the map $\iota_B \otimes Id_C$ on A to A^- , but as the latter map behaves badly with respect to products (it neither

preserves them nor inverts them), we consider only the restriction θ to A^- . We also remark that interchanging the roles of B and C just means replacing θ by $-\theta$ and inverting the sign of the Albert form.

The reduced norm $N_{\mathbb{F}}^A$ is a degree 4 form on A which takes any tensor product $b \otimes c$ to $N_{\mathbb{F}}^B(b)^2 N_{\mathbb{F}}^C(c)^2$. In certain calculations we shall need to evaluate it, which is done, under some assumptions, in the following

Proposition 5.3. *If $A = M_2(B)$ for some quaternion algebra B then the equality*

$$N_{\mathbb{F}}^A \begin{pmatrix} a & b \\ c & d \end{pmatrix} = N_{\mathbb{F}}^B(a)N_{\mathbb{F}}^B(d) + N_{\mathbb{F}}^B(b)N_{\mathbb{F}}^B(c) - \text{Tr}_{\mathbb{F}}^B(\bar{a}b\bar{d}c)$$

holds for every $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in A$, i.e., every a, b, c , and d from B .

Proof. First, the assertion holds in case one of a, b, c , or d is 0. This follows from evaluation of 4×4 determinants in case $B = M_2(\mathbb{F})$ and $A = M_4(\mathbb{F})$, hence holds in general since B may be considered as a subalgebra of matrices over a splitting field and the assertions are invariant under scalar extensions. Now, if one entry is invertible then we can determine the reduced norm by right multiplication with a matrix of the sort $\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}$ (of reduced norm 1): For example, if a is invertible then we take $x = -a^{-1}b$ and find that that $N_{\mathbb{F}}^A \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ equals $N_{\mathbb{F}}^B(a)N_{\mathbb{F}}^B(d - ca^{-1}b)$ by multiplicativity, and then using Lemma 1.1 we get the asserted value. Similar considerations cover the cases where b, c , or d are invertible, which completes the proof in the case where B is a division algebra. In case $B = M_2(\mathbb{F})$ and all of a, b, c , and d are non-zero and not invertible, we may conjugate everything in B (an operation leaving our expression invariant) such that a becomes the matrix $\begin{pmatrix} t & 0 \\ 0 & 0 \end{pmatrix}$ for some $t \in \mathbb{F}^\times$. Observe that left multiplication by $\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}$ takes the upper left entry of our matrix to $a + xc$. Recall that $\det c = 0$ but $c \neq 0$. Hence if c has right column 0 then we may choose x such that $a + xc = 0$, while if the right column of c is not 0 we may choose x such that xc has upper row 0 and lower row non-zero, so that $a + xc$ becomes invertible. As our expression is invariant under left multiplication by $\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}$, this completes the proof of the Proposition. \square

We can use Proposition 5.3 to get an explicit expression for the $N_{\mathbb{F}}^A$ in general. Writing an element of A as $\alpha + \delta i + \beta j + \gamma ij$ where i and j generate C , anti-commute, and square to η and ε respectively, we may embed C into $M_2(\mathbb{K})$, where $\mathbb{K} = \mathbb{F}(\sqrt{\eta})$ with Galois automorphism σ , as the algebra $(\mathbb{K}, \sigma, \delta)$, and then our element becomes an element of $A_{\mathbb{K}} = M_2(B_{\mathbb{K}})$ for which we may apply Proposition 5.3. The result is

$$\begin{aligned} N_{\mathbb{F}}^A(\alpha + \delta i + \beta j + \gamma ij) &= N_{\mathbb{F}}^B(\alpha)^2 + \eta^2 N_{\mathbb{F}}^B(\delta)^2 + \varepsilon^2 N_{\mathbb{F}}^B(\beta)^2 + \eta^2 \varepsilon^2 N_{\mathbb{F}}^B(\gamma)^2 + \\ &\quad - \eta \text{Tr}_{\mathbb{F}}^B((\bar{\alpha}\delta)^2) - \varepsilon \text{Tr}_{\mathbb{F}}^B((\bar{\alpha}\beta)^2) + \eta \varepsilon \text{Tr}_{\mathbb{F}}^B((\bar{\alpha}\gamma)^2) + \eta \varepsilon \text{Tr}_{\mathbb{F}}^B((\bar{\delta}\beta)^2) + \\ &\quad - \eta^2 \varepsilon \text{Tr}_{\mathbb{F}}^B((\bar{\delta}\gamma)^2) - \eta \varepsilon^2 \text{Tr}_{\mathbb{F}}^B((\bar{\beta}\gamma)^2) - 2\eta \varepsilon \text{Tr}_{\mathbb{F}}^B(\bar{\alpha}\beta\bar{\delta}\gamma) + 2\eta \varepsilon \text{Tr}_{\mathbb{F}}^B(\bar{\alpha}\gamma\bar{\delta}\beta), \end{aligned}$$

and it is seen to reduce to $N_{\mathbb{F}}^B(b)^2 N_{\mathbb{F}}^C(c)^2$ when our element is a single tensor $b \otimes c$ (by choosing i to be the traceless part of c if it is non-zero). More importantly, we have

Corollary 5.4. *The equality $N_{\mathbb{F}}^A(u) = |u|^4$ holds for every $u \in A^-$.*

Proof. One way to see it is by writing $u = x \otimes 1 + 1 \otimes y$ and choosing $i = y$ in the basis for C for the latter formula (if it does not vanish). Alternatively, we consider $A = M_2(B)$ (with $C = M_2(\mathbb{F})$) first, where elements of $A^- = M_2(B)^-$ take the form $u = \begin{pmatrix} \lambda & -r \\ s & -\bar{\lambda} \end{pmatrix}$, in which $\lambda \in B$ and r and s are from \mathbb{F} . For such an element we have $|u|^2 = N_{\mathbb{F}}^B(\lambda) - rs$ (this expression resembles the Moore determinant of Hermitian matrices—see Corollary 5.10 below), and the expression from Proposition 5.3 indeed yields the square of the latter expression. For the general case we embed C in $M_2(\mathbb{K})$ and A in $M_2(B_{\mathbb{K}})$ as above and use extension of scalars. This proves the corollary. \square

Corollary 5.4 emphasizes the fact that an element $u \in A^-$ is invertible if and only if $|u|^2 \neq 0$. We also note that for every $a \in A$ we have $N_{\mathbb{F}}^A(a) = N_{\mathbb{F}}^A(\bar{a})$, either by a direct evaluation using our formulae (and the fact that the same assertion holds for the reduced norms of the quaternion algebras B and C) or by Corollary 2.2 of [KMRT].

The bi-quaternion algebra A operates on itself via $g : M \mapsto gM\bar{g}$, and this action preserves the subspaces A^{\pm} . The properties for this action of A on the 6-dimensional space A^- underlying the Albert form which will be useful for our purposes are given in the following

Proposition 5.5. *The action of $g \in A$ multiplies the norm $|u|^2$ of the element $u \in A^-$ by $N_{\mathbb{F}}^A(g)$. The only invertible elements whose action on A^- is a global scalar multiplication are scalars from \mathbb{F}^{\times} .*

Proof. We first consider the case where $C = M_2(\mathbb{F})$ and $A = M_2(B)$. The element u takes the form $\begin{pmatrix} \lambda & -r \\ s & -\bar{\lambda} \end{pmatrix}$, with $|u|^2 = N_{\mathbb{F}}^B(\lambda) - rs$, as in the proof of Corollary 5.3. Now, for $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M_2(B)$ we have $\bar{g} = \begin{pmatrix} \bar{d} & -\bar{b} \\ -\bar{c} & \bar{a} \end{pmatrix}$, and then $gu\bar{g} \in M_2(B)^-$ involves the quaternion $a\lambda\bar{d} + sb\bar{d} + ra\bar{c} + b\bar{\lambda}\bar{c}$ and the numbers $rN_{\mathbb{F}}^B(a) + Tr_{\mathbb{F}}^B(a\lambda\bar{b}) + sN_{\mathbb{F}}^B(b)$ and $sN_{\mathbb{F}}^B(d) + Tr_{\mathbb{F}}^B(c\lambda\bar{d}) + rN_{\mathbb{F}}^B(c)$ in the places of r and s respectively. Evaluating the norm of the element involving these parameters yields the desired expression

$$[N_{\mathbb{F}}^B(a)N_{\mathbb{F}}^B(d) + N_{\mathbb{F}}^B(b)N_{\mathbb{F}}^B(c) - Tr_{\mathbb{F}}^B(\bar{a}b\bar{d}c)](N_{\mathbb{F}}^B(\lambda) - rs).$$

In addition, if g is such that these parameters are multiples of the original ones by a global scalar, then $b\bar{d} = a\bar{c} = 0$ and $N_{\mathbb{F}}^B(b) = N_{\mathbb{F}}^B(c) = 0$, and the fact that the trace form on B is non-degenerate implies also $\bar{b}a = \bar{d}c = 0$. Since invertible elements of A have non-zero norm, we find that $N_{\mathbb{F}}^B(a)$ and $N_{\mathbb{F}}^B(d)$ must not vanish, hence a and d are invertible and thus $b = c = 0$. The global scalar property now implies $N_{\mathbb{F}}^B(a) = N_{\mathbb{F}}^B(d)$, examining the effect on λ being d or \bar{a} implies $a = d$, and the scalar multiplication property shows that this element $a = d$ is central in A . Hence $g \in \mathbb{F}^{\times}$ as desired, and the converse direction is trivial. In the case where A is a division algebra, we can extend scalars to a splitting field \mathbb{K} of C , apply the arguments over \mathbb{K} , and then return to our space over \mathbb{F} . This works since the multiplier lies in \mathbb{F} , it is the reduced norm of

an element of a bi-quaternion algebra over \mathbb{F} , and the only elements in A which become scalars in $A_{\mathbb{K}}$ come from \mathbb{F} . This proves the proposition. \square

Before we define the group which will become the Gspin group in this case, we need another

Lemma 5.6. *The equality $\widetilde{gu\bar{g}} = N_{\mathbb{F}}^A(g)\bar{g}^{-1}\tilde{u}g^{-1}$ holds for any $g \in A^\times$ and $u \in A^-$.*

Proof. From Lemma 5.2 and Proposition 5.5 we get

$$gu\bar{g} \cdot \widetilde{gu\bar{g}} = |\widetilde{gu\bar{g}}|^2 = N_{\mathbb{F}}^A(g)|u|^2 = N_{\mathbb{F}}^A(g)g \cdot u\tilde{u} \cdot g^{-1} = N_{\mathbb{F}}^A(g)gu\bar{g} \cdot \bar{g}^{-1}\tilde{u}g^{-1},$$

since $|u|^2 = u\tilde{u}$ is a scalar (hence central in A). The assertion now follows for every $u \in A^- \cap A^\times$, and extends to all of A^- by linearity and the fact that A^- admits a basis of (orthogonal) vectors which lie in A^\times . This proves the lemma. \square

We consider the subgroup $A^{(\mathbb{F}^\times)^2}$ of A^\times . This group contains every single tensor $b \otimes c$ with $b \in B^\times$ and $c \in C^\times$ (and in particular any scalar from \mathbb{F}^\times), and Corollary 5.4 shows that $A^- \cap A^\times \subseteq A^{(\mathbb{F}^\times)^2}$. Let $\tilde{A}^{(\mathbb{F}^\times)^2}$ denote the “metaplectic-like” double cover of $A^{(\mathbb{F}^\times)^2}$ consisting of pairs $(g, t) \in A^{(\mathbb{F}^\times)^2} \times \mathbb{F}^\times$ such that $N_{\mathbb{F}}^A(g) = t^2$ (with coordinate-wise product). This group appears in Section 17 of [KMRT] in relation with the orthogonal group of $O(V)$, and we prove this connection using very simple means.

Lemma 5.7. *The operation $(g, t) : u \mapsto \frac{gu\bar{g}}{t}$ defines a map $\tilde{A}^{(\mathbb{F}^\times)^2} \rightarrow O(A^-)$, with kernel consisting of the elements (r, r^2) for $r \in \mathbb{F}^\times$. The automorphism $g \mapsto \bar{g}^{-1}$ of A^\times preserves $A^{(\mathbb{F}^\times)^2}$, and $(g, t) \mapsto (t\bar{g}^{-1}, t)$ is an automorphism of $\tilde{A}^{(\mathbb{F}^\times)^2}$ of order 2. If $\tilde{\theta}$ generates a cyclic group of order 2 acting on $\tilde{A}^{(\mathbb{F}^\times)^2}$ by this automorphism then sending it to θ yields a map from the associated semi-direct product to $O(A^-)$.*

Proof. The fact that the image of $\tilde{A}^{(\mathbb{F}^\times)^2}$ lies in $O(A^-)$ follows from Proposition 5.5 and the definition of $\tilde{A}^{(\mathbb{F}^\times)^2}$. As for the kernel, Proposition 5.5 implies that elements of the kernel must lie over \mathbb{F}^\times , and the fact that they take the form (r, r^2) (and that these elements indeed lie in $\tilde{A}^{(\mathbb{F}^\times)^2}$) is immediate. The fact that $N_{\mathbb{F}}^A(\bar{g}^{-1})$ is the reciprocal of $N_{\mathbb{F}}^A(g)$ shows that the automorphism preserves $A^{(\mathbb{F}^\times)^2}$, and the fact that $N_{\mathbb{F}}^A(t) = t^4$ and $t\bar{t}^{-1} = 1$ for $t \in \mathbb{F}^\times$ proves the assertions about the automorphism of $\tilde{A}^{(\mathbb{F}^\times)^2}$. As $\theta \in O(A^-)$ (clear), the result about the semi-direct product follows from Lemma 5.6 and the fact that $t^2 = N_{\mathbb{F}}^A(g)$ for $(g, t) \in \tilde{A}^{(\mathbb{F}^\times)^2}$. This proves the lemma. \square

The multiplication by t in the automorphism of $\tilde{A}^{(\mathbb{F}^\times)^2}$ is not necessary. However, we put it there for certain maps below (see Lemmas 7.3 and 11.6) to take a neater form.

As in the previous cases, we now prove

Lemma 5.8. *The reflection in the vector $g \in A^- \cap A^\times$ lies in the image of the semi-direct product from Lemma 5.7, and takes the form $u \mapsto \frac{g\bar{u}g}{|g|^2}$.*

Proof. Corollary 5.4 shows that $(g, |g|^2) \in \tilde{A}^{(\mathbb{F}^\times)^2}$, so that the asserted operation lies in the image of the map from Lemma 5.7. Now, $\bar{g} = -g$ since $g \in A^-$, and Lemma 5.2 shows that the denominator is $g\bar{g}$. In the same manner as in the previous cases, replacing $-g\bar{u}$ by $u\bar{g}$ for $u \in g^\perp$ and just substituting g for $u = g$ shows that this expression yields u for $u \in g^\perp$ case and $-g = -u$ for $u = g$. This proves the lemma. \square

We remark that the operation from Lemma 5.8 is indeed invariant under interchanging the roles of B and C , since then both θ and $|g|^2$ are being inverted.

We now come to prove

Theorem 5.9. *The Gspin group of A^- is $\tilde{A}^{(\mathbb{F}^\times)^2}$, and it is generated by the elements $(g, |g|^2)$ with $g \in A^- \cap A^\times$. The spin group is a subgroup mapping bijectively onto A^1 .*

Proof. As in the previous cases, the fact that θ has determinant -1 (it inverts a 3-dimensional subspace), Lemma 5.8, and Proposition 2.2 show that the map from the semi-direct product in Lemma 5.7 to $O(A^-)$ is surjective, and its restriction to $\tilde{A}^{(\mathbb{F}^\times)^2}$ maps surjectively onto $SO(A^-)$, with kernel (isomorphic to) \mathbb{F}^\times . Hence $Gspin(A^-) = \tilde{A}^{(\mathbb{F}^\times)^2}$, and from the structure of the semi-direct product we deduce the generation of the latter group by elements arising from $A^- \cap A^\times$. Note that here the latter assertion is not so easy to establish in a direct manner. As θ has spinor norm 1 (the 3-dimensional space which it inverts has determinant 1), Lemma 5.8 implies that the spinor norm of the element $(g, |g|^2 = t)$ is $t = |g|^2$. As these elements generate $\tilde{A}^{(\mathbb{F}^\times)^2}$, multiplicativity shows that the spinor norm of every element $(g, t) \in \tilde{A}^{(\mathbb{F}^\times)^2}$ is just t (in particular, the element $(1, -1)$, operating as $-Id_{A^-}$, has the required spinor norm -1 like the determinant of A^-). Thus, the elements lying over $SO^1(A^-)$ come with second coordinate from $(\mathbb{F}^\times)^2$, and as (r, r^2) operates trivially for every $r \in \mathbb{F}^\times$, we may divide by it and consider just elements of $\tilde{A}^{(\mathbb{F}^\times)^2}$ of the form $(g, 1)$. As this group maps isomorphically onto A^1 in the natural projection $\tilde{A}^{(\mathbb{F}^\times)^2} \rightarrow A^{(\mathbb{F}^\times)^2}$, and its intersection with the image of the kernel \mathbb{F}^\times is just ± 1 , we find that the spin group $spin(A^-)$ is just (this isomorphic image of) A^1 . This proves the theorem. \square

Observe that while the space depends on the choice of the involution on A (i.e., the decomposition as a tensor product of quaternion algebras), the groups $SO(A^-)$ and $SO^1(A^-)$ indeed do not.

For the isotropic case, we have

Corollary 5.10. *If A^- is isotropic then there is a quaternion algebra B such that the Gspin group is the double cover of the group $\widetilde{GL}_2^{(\mathbb{F}^\times)^2}(B)$ consisting of those matrices in which the expression from Proposition 5.5 is a square, where*

the double cover involves choosing a square root for it. The spin group is just the subgroup $GL_2^1(B)$ in which this expression equals 1. If A^- splits another hyperbolic plane then it is the sum of three such planes, the Gspin group is $\widetilde{GL}_4^{(\mathbb{F}^\times)^2}(\mathbb{F})$, and the spin group is $SL_4(\mathbb{F})$.

Proof. The fact that A cannot be a division algebra follows immediately from the fact that some elements of A^- are zero-divisors when A^- is isotropic. Moreover, in this case A^- splits a hyperbolic plane, so that by choosing the subspace to normalize as the B_0 part to be a subspace of the orthogonal complement of this hyperbolic plane we may assume (see Corollary 3.10) that $C = M_2(\mathbb{F})$. Hence $A = M_2(B)$, the space A^- consists of the matrices $\begin{pmatrix} \lambda & -r \\ s & -\lambda \end{pmatrix}$ from the proof of Corollary 5.4, and $A^{(\mathbb{F}^\times)^2} = GL_2^{(\mathbb{F}^\times)^2}(B)$ is defined according to the expression from Proposition 5.5. Thus, Theorem 5.9 shows that $Gspin(M_2(B)^-)$ is the double cover $\widetilde{GL}_2^{(\mathbb{F}^\times)^2}(B)$, and $spin(M_2(B)^-)$ is $A^1 = GL_2^1(B)$ according to the same expression. Observe that the complement of a hyperbolic plane (the r and s coordinates) is the space B from Corollary 4.5. Hence A^- splits another hyperbolic plane if and only if B also splits (see Corollary 4.6), and as in this case the complement of these two planes has discriminant 1, it is again a hyperbolic plane by Corollary 3.5. Hence $A = M_4(\mathbb{F})$, the reduced norm is the determinant, and the Gspin and spin groups from Theorem 5.9 become indeed $\widetilde{GL}_4^{(\mathbb{F}^\times)^2}(\mathbb{F})$ and $SL_4(\mathbb{F})$ respectively. This proves the corollary. \square

Theorem 16.5 of [KMRT] relates the index of A (4 in case of a division algebra, 1 in the split case, 2 in the middle) to the number of hyperbolic planes one can split from the Albert form of A (0, 3, and 1 respectively). The hardest direction is to prove that if the Albert form is anisotropic then A is a division algebra, yielding the inverse direction of Corollary 5.10. On the other hand, all the remaining assertions are either trivial or follow from the proof of Corollary 5.10.

We also have

Corollary 5.11. *Given a quaternion algebra B , the groups $\widetilde{GL}_2^{(\mathbb{F}^\times)^2}(B)$ and $GL_2^1(B)$ are also the Gspin and spin groups of the space $M_2^{Her}(B)$ of Hermitian 2×2 matrices over B , which is quadratic with the vector norm being (minus) the Moore determinant, through the action via $g : X \mapsto \frac{gX\iota_B(g)^t}{t}$. In addition, the groups $\widetilde{GL}_4^{(\mathbb{F}^\times)^2}(\mathbb{F})$ and $SL_4(\mathbb{F})$ are the Gspin and spin groups of $M_4^{as}(\mathbb{F})$ the space of anti-symmetric matrices over \mathbb{F} (with minus the Pfaffian as the quadratic form) on which they operate as $g : T \mapsto \frac{gTg^t}{t}$, as well as on another space, with the operation $(g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, t) : S \mapsto \begin{pmatrix} a & b \\ c & d \end{pmatrix} S \begin{pmatrix} d^t & -b^t \\ -c^t & a^t \end{pmatrix} / t$.*

Proof. The first assertion follows, as in Corollaries 3.11 and 4.7, from Corollary 5.10, Lemma 1.3, and the fact that $M_2^{Her}(B)$ is the image of $M_2(B)^-$ under right multiplication by $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$. Indeed, the expression $N_{\mathbb{F}}^B(\lambda) - rs$ becomes minus the Moore determinant of $\begin{pmatrix} r & \lambda \\ \lambda & s \end{pmatrix}$. For the second assertion, we may apply

Lemma 1.3 again, but now for the elements of $B = M_2(\mathbb{F})$ which are entries of matrices in $A = M_2(B)$. The representation just described becomes $M_4^{as}(\mathbb{F})$ under this operation: The diagonal blocks rI and sI are taken to multiples of the anti-symmetric matrix $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, the off-diagonal blocks are taken care of by Lemma 1.3 and the fact that $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ is anti-symmetric, and the Moore determinant becomes the Pfaffian. The fourth representation is obtained from the one considered in Theorem 5.9 by this right multiplication. This proves the corollary. \square

The representation on $M_4^{as}(\mathbb{F})$ uses an orthogonal involution, while the two others use symplectic ones (compare with Proposition 2.23(1) of [KMRT] again). The generator θ of $O(A^-)/SO(A^-)$ corresponds to the adjoint involution of 2×2 matrices in Theorem 5.9 in case $A = M_2(B)$, and to $X \mapsto -X^t$ on the space $M_2^{Her}(B)$ from Corollary 5.11. On the other hand, conjugating by $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ sends the latter involution to $X \mapsto -adj X$ on $M_2^{Her}(B)$, and the product of X and its image under this involution yields $|X|^2$ again. θ yields involutions on $M_4^{as}(\mathbb{F})$ and on the other space as well, and after appropriate conjugations (by $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ on the entries as 2×2 matrices, or on both), we see that these spaces also admit involutions with the property that the product of a vector with its image yields the vector norm. The latter involution on $M_4^{as}(\mathbb{F})$ will be denoted $T \mapsto \bar{T}$.

6 Dimension 5

The spaces we get in dimension 5 are those given in

Lemma 6.1. *Every 5-dimensional space is isometric, up to a scalar multiple, to the orthogonal complement of an anisotropic vector Q in the space A^- arising from a bi-quaternion algebra A presented as $B \otimes C$.*

Proof. First we remark that for anisotropic $Q \in A^-$, the subspace Q^\perp of A^- is 5-dimensional and non-degenerate. Conversely, a 5-dimensional quadratic space can be extended, uniquely up to the choice of a generator, to a 6-dimensional space of discriminant 1: If the space has some discriminant (or determinant) d , this is done by adding an element Q with $|Q|^2 = -d$. The lemma now follows directly from Lemma 5.1. \square

In fact, by choosing the 3-dimensional subspace from the proof of Lemma 5.1 to be contained in our original space Q^\perp we make sure that $Q \in 1 \otimes C_0$ (see Proposition 6.5 below for a more precise statement), but we shall not need this fact. In any case, we shall write our vector space as $Q^\perp \subseteq A^-$. The next step is

Lemma 6.2. *The groups $SO(Q^\perp \subseteq A^-)$, $SO^1(Q^\perp \subseteq A^-)$, $Gspin(Q^\perp \subseteq A^-)$, and $spin(Q^\perp \subseteq A^-)$ are the subgroups of $SO(A^-)$, $SO^1(A^-)$, $Gspin(A^-)$, and $spin(A^-)$ respectively, consisting of those elements whose action stabilizes Q .*

Proof. The Witt Cancellation Theorem implies that any element of $O(Q^\perp \subseteq A^-)$ comes from an element of $O(A^-)$ under which Q^\perp is invariant, and $\mathbb{F}Q$ must

therefore also be invariant under such extensions. As $O(\mathbb{F}) = \{\pm 1\}$, and -1 has determinant -1 there, the assertion about $SO(Q^\perp \subseteq A^-)$ follows. Considering extensions of $O(Q^\perp \subseteq A^-)$ to $O(A^-)$ by taking only $+1$ on $\mathbb{F}Q$, Proposition 2.2 shows that any element there has the same spinor norm in both $O(Q^\perp \subseteq A^-)$ and $O(A^-)$ (since this is true for reflections in vectors from Q^\perp). The assertion for $SO^1(Q^\perp \subseteq A^-)$ follows, and for the remaining two groups we just use the maps into $O(A^-)$. This proves the lemma. \square

For a simpler description, consider the group $A_{\mathbb{F}Q}^\times$ of those $g \in A^\times$ such that $gQ\bar{g} \in \mathbb{F}Q$ (i.e., those which preserve the 1-dimensional space $\mathbb{F}Q$), which comes with a group homomorphism $m : A_{\mathbb{F}Q}^\times \rightarrow \mathbb{F}^\times$ defined by $gQ\bar{g} = t(g)Q$. We remark that $A_{\mathbb{F}Q}^\times$ can be seen as the group of similitudes of A with respect to the involution $x \mapsto Q\bar{x}Q^{-1}$ (which is symplectic since $Q \in A^-$). We now have

Lemma 6.3. *The group $A_{\mathbb{F}Q}^\times$ is a subgroup of $A^{(\mathbb{F}^\times)^2}$. The double cover $\tilde{A}^{(\mathbb{F}^\times)^2}$ splits over $A_{\mathbb{F}Q}^\times$, and the splitting group contains the group $\{(r, r^2) | r \in \mathbb{F}^\times\}$.*

Proof. The equality $gQ\bar{g} = t(g)Q$ holding for $g \in A_{\mathbb{F}Q}^\times$ implies, by Proposition 5.5 and the fact that $|Q|^2 \neq 0$, the equality $N_{\mathbb{F}}^A(g) = t(g)^2$. This establishes the first assertion, as well as introducing the splitting homomorphism $g \mapsto (g, t(g))$ from $A_{\mathbb{F}Q}^\times$ into $\tilde{A}^{(\mathbb{F}^\times)}$. The fact that $rQ\bar{r} = r^2Q$ for $r \in \mathbb{F}^\times$ completes the proof of the lemma. \square

We can now prove

Theorem 6.4. *We have $Gspin(Q^\perp \subseteq A^-) = A_{\mathbb{F}Q}^\times$, and $spin(Q^\perp \subseteq A^-)$ is the group A_Q^1 of elements of $g \in A^1$ such that $gQ\bar{g} = Q$.*

Proof. For the Gspin group, Lemma 6.2 shows that it suffices to prove that the image of $A_{\mathbb{F}Q}^\times$ under the splitting map from Lemma 6.3 is the stabilizer of Q under the action of $\tilde{A}^{(\mathbb{F}^\times)}$. But it is clear from the definition that the image of this splitting map stabilizes Q , and that elements of this stabilizer must come from $A_{\mathbb{F}Q}^\times$. As replacing m by $-m$ yields an element taking Q to $-Q$, the assertion for the Gspin group follows. The statement for the spin group follows directly from Lemma 6.2, since no scalar appears in the action of $(g, 1)$ for A^1 . Note that for $g \in A_Q^1$ we have $t(g) = 1$, so that the corresponding element is indeed of the form $(g, 1)$ for A^1 . This proves the proposition. \square

Note that our construction is based on the choice of $Q \in A^-$. However, the only parameter which is required to know the isomorphism class of $A_{\mathbb{F}Q}^\times$ and A_Q^1 is given in the following

Proposition 6.5. *If an element $R \in A^-$ satisfies $|R|^2 \in |Q|^2(\mathbb{F}^\times)^2 N_{\mathbb{F}}^A(A^\times)$ then $A_{\mathbb{F}Q}^\times \cong A_{\mathbb{F}R}^\times$ and $A_Q^1 \cong A_R^1$.*

Proof. First note that $A_{\mathbb{F}rQ}^\times = A_{\mathbb{F}Q}^\times$ and $A_{rQ}^1 = A_Q^1$ for every $r \in \mathbb{F}^\times$, so once only (non-zero) vector norms are involved, we can consider them modulo $(\mathbb{F}^\times)^2$. Now, for any $a \in A^\times$, conjugation by a takes $A_{\mathbb{F}Q}^\times$ to $A_{\mathbb{F}aQ\bar{a}}^\times$ and A_Q^1 to $A_{aQ\bar{a}}^1$. Applying this to (a, t) with $a \in A^{(\mathbb{F}^\times)^2}$ and using the transitivity of the action of $SO(A^-)$ on elements of the same vector norm (Witt Cancellation again) establishes the assertion in case $|R|^2 = |Q|^2$ (hence also if $|R|^2 \in |Q|^2(\mathbb{F}^\times)^2$), and then doing so for general $a \in A^\times$ allows us to divide also by $N_{\mathbb{F}}^A(A^\times)$. This proves the proposition. \square

When we consider the case where A^- is isotropic, the relation from Proposition 6.5 becomes simpler by the following

Lemma 6.6. *If $A = M_2(B)$ then $N_{\mathbb{F}}^A(A) = N_{\mathbb{F}}^B(B)$ and $N_{\mathbb{F}}^A(A^\times) = N_{\mathbb{F}}^B(B^\times)$.*

Proof. For any $b \in B$ we get $N_{\mathbb{F}}^B(b)$ as the reduced norm of the element $\begin{pmatrix} b & 0 \\ 0 & 1 \end{pmatrix}$ of A . For the other direction, the proof of Proposition 5.5 shows that if a matrix in $M_2(B)$ has an invertible entry then its reduced norm is the product of two norms from B (e.g., $N_{\mathbb{F}}^B(a)N_{\mathbb{F}}^B(d - ca^{-1}b)$ if $a \in B^\times$), which completes the proof for division algebras B since $N_{\mathbb{F}}^B$ is multiplicative. As for $B = M_2(\mathbb{F})$ and $A = M_4(\mathbb{F})$, every element of \mathbb{F} is both a 2×2 determinant and a 2×2 determinant, the proof of the lemma is complete. \square

Rather than isotropy, in this case we have

Corollary 6.7. *If the 5-dimensional space of discriminant d represents $-d$ then its Gspin and spin groups are isomorphic to $GSp_B\begin{pmatrix} -\delta & 0 \\ 0 & 1 \end{pmatrix}$ and $Sp_B\begin{pmatrix} -\delta & 0 \\ 0 & 1 \end{pmatrix}$ respectively, for some quaternion algebra B and some $\delta \in \mathbb{F}^\times$ which may be taken from a set of representatives for $F^\times/N_{\mathbb{F}}^B(B^\times)$. In case this space splits two hyperbolic planes, these groups are isomorphic to the classical groups $GSp_4(\mathbb{F})$ and $Sp_4(\mathbb{F})$ respectively.*

Proof. In this case (which covers the case of an isotropic space since a hyperbolic plane represents every element of \mathbb{F}) the space A^- from the proof of Lemma 6.1 will be isotropic. Hence we can assume that $A = M_2(B)$ (with the associated involution) by Corollary 5.10, and the Gspin and spin groups from Theorem 6.4 are $GL_2(B)_{\mathbb{F}Q}$ and $GL_2^1(B)_Q$ respectively. Lemma 6.5 shows that the only parameter required for determining the isomorphism classes of these groups is $|Q|^2$ up to $(\mathbb{F}^\times)^2 N_{\mathbb{F}}^A(A^\times)$, and by Lemma 6.6 and the fact that $(\mathbb{F}^\times)^2 \subseteq N_{\mathbb{F}}^B(B^\times)$ for quaternion algebras (as $N_{\mathbb{F}}^B(r) = r^2$), we may take the vector norm δ from the required set of representatives and any Q with $|Q|^2 = \delta$ will do. We choose Q to be the element $\begin{pmatrix} 0 & \delta \\ 1 & 0 \end{pmatrix}$ of $M_2(\mathbb{F})_0 = 1 \otimes C_0 \subseteq M_2(B)^-$, with $|Q|^2 = -\det Q = \delta$. Using Corollary 5.11, the fact that multiplying Q by $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ from the right gives this diagonal matrix shows that our groups are indeed $GSp_B\begin{pmatrix} -\delta & 0 \\ 0 & 1 \end{pmatrix}$ and $Sp_B\begin{pmatrix} -\delta & 0 \\ 0 & 1 \end{pmatrix}$.

In case our space contains two hyperbolic planes, Corollary 5.10 shows that B also splits. As $N_{\mathbb{F}}^B(B^\times) = \mathbb{F}^\times$ in this case, we have, up to isomorphism, the same group for every choice of Q (Lemma 6.5). Instead of taking Q as

above, we recall from the proof of Corollary 5.4 that the space A^- (now for split B and C) consists of matrices of the form $\begin{pmatrix} Y & -rI \\ sI & -adj Y \end{pmatrix}$ with r and s from \mathbb{F} and $Y \in M_2(\mathbb{F})$, and we take Q to be the element with $r = s = 0$ and $Y = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$. Going over to the representation from Corollary 5.10 in which $\widehat{GL_4^{(\mathbb{F}^\times)^2}}(\mathbb{F})$ operates by $g : T \mapsto \frac{gTg^t}{t}$, our groups are easily seen to be the classical $GSp_4(\mathbb{F})$ and $Sp_4(\mathbb{F})$. This proves the corollary. \square

As for equivalent representations, we get

Corollary 6.8. *The groups $GSp_B\begin{pmatrix} -\delta & 0 \\ 0 & 1 \end{pmatrix}$ and $Sp_B\begin{pmatrix} -\delta & 0 \\ 0 & 1 \end{pmatrix}$ are the Gspin and spin groups of the orthogonal complement of $\begin{pmatrix} -\delta & 0 \\ 0 & 1 \end{pmatrix}$ inside $M_2^{Her}(B)$ from Corollary 5.11. The classical groups $GSp_4(\mathbb{F})$ and $Sp_4(\mathbb{F})$ can be seen as the Gspin and spin group of either the complement of $\begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix}$ in $M_4^{as}(\mathbb{F})$ or of the orthogonal complement of the adjoint representation on the Lie algebra $\mathfrak{gsp}_4(\mathbb{F})$ of $GSp_4(\mathbb{F})$ inside $M_4(\mathbb{F}) = \mathfrak{gl}_4(\mathbb{F})$.*

Proof. This follows directly by restricting the representations from Corollary 5.11 to our groups. Note that the operation $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} : S \mapsto \begin{pmatrix} a & b \\ c & d \end{pmatrix} S \begin{pmatrix} d^t & -b^t \\ -c^t & a^t \end{pmatrix}$ is conjugation for $Sp_4(\mathbb{F})$ (and conjugation tensored with the determinant for $GSp_4(\mathbb{F})$), leaving the Lie algebra $\mathfrak{sp}_4(\mathbb{F})$ invariant. The additional invariant vector is I , adding which to $\mathfrak{sp}_4(\mathbb{F})$ yields $\mathfrak{gsp}_4(\mathbb{F})$. This proves the corollary. \square

We remark that the simplicity of $Sp_4(\mathbb{F})$ as an algebraic group implies that the action on $\mathfrak{sp}_4(\mathbb{F})$ is an irreducible representation of $Sp_4(\mathbb{F})$. Hence Corollary 6.8 yields a complete reduction of the representation $\mathfrak{gl}_4(\mathbb{F})$ of these groups as the direct sum of $\mathfrak{sp}_4(\mathbb{F})$, $\mathbb{F}I$ (as the determinant), and the 5-dimensional representation of $Sp_4(\mathbb{F})$ as an SO or SO^1 group. The adjoint representation appears for the group $Sp_B\begin{pmatrix} -\delta & 0 \\ 0 & 1 \end{pmatrix}$ in Corollary 6.8 only if $\delta = -1$.

7 Dimension 6, General Discriminant

If our 6-dimensional quadratic space over \mathbb{F} now has some discriminant d , take $\mathbb{E} = \mathbb{F}(\sqrt{d})$ with Galois automorphism ρ as before. Our object of interest will be bi-quaternion algebras over \mathbb{E} , which take the form $A_{\mathbb{E}}$ for some bi-quaternion algebra A over \mathbb{F} . Given a presentation of A as $B \otimes C$ as in Lemma 5.1, both the orthogonal involution $\iota_B \otimes \iota_C \otimes Id_{\mathbb{E}} : x \mapsto \bar{x}$ and the unitary involution $\iota_B \otimes \iota_C \otimes \rho : x \mapsto \bar{x}^\rho$ are defined on $A_{\mathbb{E}}$. The space $A_{\mathbb{E}}^-$ from Lemma 5.1 is defined as a quadratic space over \mathbb{E} , and we shall consider the vector space over \mathbb{F} which is defined in the following

Lemma 7.1. *Take some anisotropic $Q \in A^-$. The set of elements $u \in A_{\mathbb{E}}^-$ satisfying the equality $u^\rho = -\frac{Q\bar{u}Q}{|Q|^2}$ form a 6-dimensional quadratic space of discriminant d over \mathbb{F} . Every quadratic space of dimension 6 and discriminant d over \mathbb{F} is isomorphic, up to rescaling, to a space which is obtained in this way.*

Proof. The proof of Lemma 5.8 shows that the expression, which we require to equal u^ρ , is just u if $u \in Q^\perp \subseteq A_{\mathbb{E}}^-$, and is $-cQ$ in case $u = cQ$. As $Q^\rho = Q$, it follows that the elements u which satisfy this property is precisely $(Q^\perp \subseteq A^-) \oplus \mathbb{E}_0 Q$, which is the quadratic space over \mathbb{F} which is obtained from A^- by rescaling the vector norm of Q by d . It is thus 6-dimensional and of the required discriminant. Conversely, given a space of dimension 6 and discriminant d , choose an arbitrary isotropic vector, and divide its vector norm by d . The resulting space has discriminant 1, so that by Lemma 5.1 it takes the form A^- for some bi-quaternion algebra $A = B \otimes C$ over \mathbb{F} . But then we may extend scalars to \mathbb{E} , and multiplying $|Q|^2$ back by d (to get its original value) means precisely replacing the \mathbb{F} -subspace A^- of $A_{\mathbb{E}}^-$ by the space $(Q^\perp \subseteq A^-) \oplus \mathbb{E}_0 Q$ considered above. This completes the proof of the lemma. \square

We denote the space from Lemma 7.1 by $(A_{\mathbb{E}}^-)_{\rho, Q}$, and observe that extending its scalars to \mathbb{E} also gives $A_{\mathbb{E}}^-$. The formulae from Lemma 5.2 hold for $u \in (A_{\mathbb{E}}^-)_{\rho, Q}$, since such elements lie in $A_{\mathbb{E}}^-$ and the expressions are \mathbb{F} -valued on $(A_{\mathbb{E}}^-)_{\rho, Q}$. Note that ρ acts on $(A_{\mathbb{E}}^-)_{\rho, Q}$ like the reflection in a generator of $\mathbb{E}_0 Q$.

The group $A_{\mathbb{E}}^\times$ operates on A either through the map from Lemma 5.7 (preserving $A_{\mathbb{E}}^-$), or via $g : M \mapsto gM\bar{g}^\rho$. We define $A_{\mathbb{E}, \rho, \mathbb{F}Q}^\times$ to be the subgroup of $A_{\mathbb{E}}^\times$ which stabilizes the 1-dimensional vector space $\mathbb{F}Q$ under the latter action (i.e., multiplies Q by a scalar), and let $t : A_{\mathbb{E}, \rho, \mathbb{F}Q}^\times \rightarrow \mathbb{F}^\times$ be the group homomorphism defined via $gQ\bar{g}^\rho = t(g)Q$. Note that as $\bar{Q}^\rho = -Q$ and $gQ\bar{g}^\rho$ lies in the same eigenspace of $\iota_B \otimes \iota_C \otimes \rho$, allowing $t(g)$ to be in \mathbb{E}^\times does not produce a larger group. This group can be seen as the group of similitudes of the unitary involution $x \mapsto Q\bar{x}^\rho Q^{-1}$. It is stable under ρ (just apply ρ to the defining equation), and its automorphism ρ commutes with the map t into \mathbb{F}^\times . The group which we shall consider here is the one appearing in the following

Lemma 7.2. *The group $A_{\mathbb{E}, \rho, \mathbb{F}Q}^{t^2}$ consisting of those elements $g \in A_{\mathbb{E}, \rho, \mathbb{F}Q}^\times$ such that $N_{\mathbb{E}}^{A_{\mathbb{E}}}(g) = t(g)^2$ has index either 1 or 2 in $A_{\mathbb{E}, \rho, \mathbb{F}Q}^\times \cap A_{\mathbb{E}}^{\mathbb{F}^\times}$. The former group is stable under ρ , and it coincides with $A_{\mathbb{E}, \rho, \mathbb{F}Q}^\times \cap A_{\mathbb{E}}^{(\mathbb{F}^\times)^2}$ unless $-1 \in (\mathbb{F}^\times)^2$ and the above index is 2.*

Proof. Given $g \in A_{\mathbb{E}, \rho, \mathbb{F}Q}^\times$, the norm $N_{\mathbb{E}}^{A_{\mathbb{E}}}(g)$ lies in $t(g)^2\mathbb{E}^1$. For $g \in A_{\mathbb{E}}^{\mathbb{F}^\times}$ the second multiplier comes from $\mathbb{E}^1 \cap \mathbb{F} = \{\pm 1\}$, so that the group $A_{\mathbb{E}, \rho, \mathbb{F}Q}^{t^2}$ equals $A_{\mathbb{E}, \rho, \mathbb{F}Q}^\times \cap A_{\mathbb{E}}^{\mathbb{F}^\times}$, unless $A_{\mathbb{E}, \rho, \mathbb{F}Q}^\times$ contains elements g with $N_{\mathbb{E}}^{A_{\mathbb{E}}}(g) = -t(g)^2$, a case in which $A_{\mathbb{E}, \rho, \mathbb{F}Q}^{t^2}$ has index 2 in this intersection. The stability under ρ follows from the fact that for $g \in A_{\mathbb{E}, \rho, \mathbb{F}Q}^{t^2}$ we have $t(g) = t(g^\rho)$ and $N_{\mathbb{F}}^A(g) = t(g)^2$ lies in \mathbb{F} . The assertion about the intersection with $A_{\mathbb{E}}^{(\mathbb{F}^\times)^2}$ follows immediately from the considerations concerning $A_{\mathbb{E}}^{\mathbb{F}^\times}$. This proves the lemma. \square

Lemma 7.2 is related to some delicate facts about the definition of GSU groups—see also Corollary 7.8 below. Note that as $N_{\mathbb{E}}^{A_{\mathbb{E}}}$ is of degree 4, Hilbert's Theorem 90 cannot help us in getting a more accurate result, like the one

following Lemma 4.2. In any case, we have $A_{\mathbb{E}, \rho, \mathbb{F}Q}^{t^2} \subseteq A_{\mathbb{E}}^{(\mathbb{F}^\times)^2} \subseteq A_{\mathbb{E}}^{(\mathbb{E}^\times)^2}$ (by Lemma 7.2), and the double cover $\tilde{A}_{\mathbb{E}}^{(\mathbb{E}^\times)^2}$ splits over $A_{\mathbb{E}, \rho, \mathbb{F}Q}^{t^2}$ —indeed, the map $g \mapsto (g, t(g))$ is, by definition, a splitting map.

Note that unless $Q^2 \in \mathbb{F}$ (i.e., $\tilde{Q} \in \mathbb{F}Q$, which means that Q lies in either $B_{\mathbb{E}} \otimes 1$ or $1 \otimes C_{\mathbb{E}}$), the map θ from Lemma 5.2 does not preserve $(A_{\mathbb{E}}^-)_{\rho, Q}$ (see Lemma 11.2 below). Moreover, the automorphism from Lemma 5.7 does not preserve $A_{\mathbb{E}, \rho, \mathbb{F}Q}^{t^2}$, since inverting Q may change the 1-dimensional space it generates. However, in order to carry out the usual considerations also in this case, we have the following

Lemma 7.3. *The element $(Q, |Q|^2)\tilde{\theta}$ squares to $(-|Q|^2, |Q|^4)$ in the semi-direct product of $\{1, \tilde{\theta}\}$ and $\tilde{A}_{\mathbb{E}}^{(\mathbb{E}^\times)^2}$. It operates on $(A_{\mathbb{E}}^-)_{\rho, Q}$ as ρ (hence preserves this space), and it takes $(g, t) \in \tilde{A}_{\mathbb{E}}^{(\mathbb{E}^\times)^2}$ to $(tQ\bar{g}^{-1}Q^{-1}, t)$ by conjugation. This is an order 2 automorphism of $\tilde{A}_{\mathbb{E}}^{(\mathbb{E}^\times)^2}$, which preserves the image of $A_{\mathbb{E}, \rho, \mathbb{F}Q}^{t^2}$ under the splitting map, and coincides with the action of ρ on the latter group.*

Proof. The product rule of the semi-direct product from Lemma 5.7 implies that the square of the element in question is $(Q \cdot |Q|^2\bar{Q}^{-1}, |Q|^2 \cdot |Q|^2)$, which equals $(-|Q|^2, |Q|^4)$ since $Q \in A_{\mathbb{E}}^-$. This element sends $u \in A_{\mathbb{E}}^-$ to $-\frac{Q\bar{u}Q}{|Q|^2}$ (use the fact that $Q \in A_{\mathbb{E}}^-$ again), which for $u \in (A_{\mathbb{E}}^-)_{\rho, Q}$ coincides with $u^\rho \in (A_{\mathbb{E}}^-)_{\rho, Q}$ according to Lemma 7.1. The formula for the conjugation follows directly from Lemma 5.7, and the fact that it is an automorphism of order 2 either follows from the centrality of $(-1, 1)$ or can be easily verified directly using the fact that $Q \in A_{\mathbb{E}}^-$ once more. Now, multiplying the equation stating that an element g lies in $A_{\mathbb{E}, \rho, \mathbb{F}Q}^\times$ by $(gQ)^{-1}$ from the left yields the equality $\bar{g}^\rho = t(g)Q^{-1}g^{-1}Q$, and after applying $\iota_B \otimes \iota_C$ we get the equality $g^\rho = t(g)Q\bar{g}^{-1}Q^{-1}$ (since $Q \in A_{\mathbb{E}}^-$). This shows that conjugation by $(Q, |Q|^2)\tilde{\theta}$ operates on $A_{\mathbb{E}, \rho, \mathbb{F}Q}^{t^2}$ as ρ , and as the latter group was seen to be preserved by ρ in Lemma 7.2, this completes the proof of the lemma. \square

We can now proceed with

Lemma 7.4. *The image of the embedding of $A_{\mathbb{E}, \rho, \mathbb{F}Q}^{t^2}$ into $\tilde{A}_{\mathbb{E}}^{(\mathbb{E}^\times)^2}$ by $g \mapsto (g, t(g))$ preserves the \mathbb{F} -subspace $(A_{\mathbb{E}}^-)_{\rho, Q}$ in the action of $\tilde{A}_{\mathbb{E}}^{(\mathbb{E}^\times)^2}$ on $A_{\mathbb{E}}^-$. Adding the element $(Q, |Q|^2)\tilde{\theta}$ from Lemma 7.3 gives a group, which contains $A_{\mathbb{E}, \rho, \mathbb{F}Q}^{t^2}$ as a subgroup of index 2, and the larger group also maps to $O((A_{\mathbb{E}}^-)_{\rho, Q})$.*

Proof. An element $u \in (A_{\mathbb{E}}^-)_{\rho, Q}$ satisfies the condition of Lemma 7.1, and for $g \in A_{\mathbb{E}, \rho, \mathbb{F}Q}^\times$ we have the formulae for g^ρ and \bar{g}^ρ from the proof of Lemma 7.3. We evaluate

$$(gu\bar{g})^\rho = t(g)Q\bar{g}^{-1}Q^{-1} \cdot \frac{-Q\bar{u}Q}{|Q|^2}t(g)Q^{-1}g^{-1}Q = -\frac{QN_{\mathbb{F}}^A(g)\bar{g}^{-1}\bar{u}g^{-1}Q}{|Q|^2}$$

(recall the $A_{\mathbb{E},\rho,\mathbb{F}Q}^{t^2}$ condition), and this equals $\frac{Q\bar{g}u\bar{g}Q}{|Q|^2}$ by Lemma 5.6 (over \mathbb{E}). This proves the first assertion. The remaining assertions follow directly from Lemma 7.3 and Lemma 5.7 over \mathbb{E} , as the quadratic structure on $(A_{\mathbb{E}}^-)_{\rho,Q}$ is defined as a subset of $A_{\mathbb{E}}^-$. This proves the lemma. \square

The issue with the reflections is a bit tricky here:

Lemma 7.5. *Let $h \in \mathbb{E}^0$ be some non-zero element (so that $d = h^2$). For any $g \in (A_{\mathbb{E}}^-)_{\rho,Q} \cap A_{\mathbb{E}}^\times$, the element ghQ^{-1} of $A_{\mathbb{E}}^\times$ lies in $A_{\mathbb{E},\rho,\mathbb{F}Q}^{t^2}$. The combination of this element with the element from Lemma 7.3 operates on $(A_{\mathbb{E}}^-)_{\rho,Q}$ as the reflection in g .*

Proof. The fact that $Q \in A_{\mathbb{E}}^-$ satisfies $Q^\rho = Q$, $g \in (A_{\mathbb{E}}^-)_{\rho,Q}$, and $h^\rho = -h$ allows us to write

$$(ghQ^{-1})Q\overline{(ghQ^{-1})}^\rho = -h^2gQ^{-1}g^\rho = d\frac{g\bar{g}Q}{|Q|^2} = \frac{d|g|^2}{|Q|^2}Q.$$

As $N_{\mathbb{F}}^A(ghQ^{-1}) = \frac{h^4N_{\mathbb{F}}^A(g)}{N_{\mathbb{F}}^A(Q)}$ equals $(\frac{d|g|^2}{|Q|^2})^2$ by Corollary 5.4, ghQ^{-1} indeed lies in $A_{\mathbb{E},\rho,\mathbb{F}Q}^{t^2}$ with $t(ghQ^{-1}) = \frac{d|g|^2}{|Q|^2}$. We may decompose the resulting element of $\tilde{A}_{\mathbb{E}}^{(\mathbb{E}^\times)^2}$ as the product of $(g, |g|^2)$, $(h, d) = (h, h^2)$ (which acts trivially on $A_{\mathbb{E}}^-$), and $(Q^{-1}, \frac{1}{|Q|^2})$. In the product with the element from Lemma 7.3 the two terms involving Q cancel, and the total action (on $A_{\mathbb{E}}^-$) is as $u \mapsto \frac{g\bar{u}\bar{g}}{|g|^2}$, which was seen in Lemma 5.8 to be the reflection in g . As $(A_{\mathbb{E}}^-)_{\rho,Q}$ inherits its quadratic structure from $A_{\mathbb{E}}^-$ and g lies in the smaller space, this total operation is the reflection in g also on $(A_{\mathbb{E}}^-)_{\rho,Q}$. This proves the lemma. \square

We are now in position to prove

Theorem 7.6. *The Gspin group $Gspin((A_{\mathbb{E}}^-)_{\rho,Q})$ is $A_{\mathbb{E},\rho,\mathbb{F}Q}^{t^2}$. The spin group $spin((A_{\mathbb{E}}^-)_{\rho,Q})$ is the subgroup $A_{\mathbb{E},\rho,Q}^1$ consisting of those elements $g \in A_{\mathbb{E}}^1$ satisfying $gQ\bar{g}^\rho = Q$.*

Proof. Theorem 5.9 shows that the semi-direct product of $\{1, \tilde{\theta}\}$ with $\tilde{A}_{\mathbb{E}}^{(\mathbb{E}^\times)^2}$, which is also generated by $(Q, |Q|^2)\tilde{\theta}$ and the latter group, maps surjectively onto $O(A_{\mathbb{E}}^-)$, with kernel \mathbb{E}^\times , and such that $\tilde{A}_{\mathbb{E}}^{(\mathbb{E}^\times)^2}$ is the inverse image of $O(A_{\mathbb{E}}^-)$. Lemma 7.4 implies that the subgroup generated by $(Q, |Q|^2)\tilde{\theta}$ and the image of $A_{\mathbb{E},\rho,\mathbb{F}Q}^{t^2}$ under $g \mapsto (g, t(g))$ lies in the inverse image of the subgroup $O((A_{\mathbb{E}}^-)_{\rho,Q})$ of $O(A_{\mathbb{E}}^-)$ (respecting the \mathbb{F} -structure), and Theorem 5.9 and the fact the elements of $O((A_{\mathbb{E}}^-)_{\rho,Q})$ have the same determinant in $O((A_{\mathbb{E}}^-)_{\rho,Q})$ and in $O(A_{\mathbb{E}}^-)$ (this is just extension of scalars) show that the inverse image of $SO((A_{\mathbb{E}}^-)_{\rho,Q})$ in the smaller group is $A_{\mathbb{E},\rho,\mathbb{F}Q}^{t^2}$. Lemma 7.5 and Proposition 2.2 imply that the map from the group generated by $A_{\mathbb{E},\rho,\mathbb{F}Q}^{t^2}$ and $(Q, |Q|^2)\tilde{\theta}$ to $O((A_{\mathbb{E}}^-)_{\rho,Q})$ is surjective. Hence the map $A_{\mathbb{E},\rho,\mathbb{F}Q}^{t^2} \rightarrow SO((A_{\mathbb{E}}^-)_{\rho,Q})$ also surjects,

and its kernel, which consists of those scalars $r \in \mathbb{E}^\times$ such that $rQ\bar{r}^\rho = r^2Q$, is precisely \mathbb{F}^\times . Note that elements $r \in \mathbb{E}_0$ also lie in $A_{\mathbb{E},\rho,\mathbb{F}Q}^{t^2}$, but as they satisfy $t(r) = -r^2$ they are not in the kernel (they operate as $-Id_{(A_{\mathbb{E}}^-)_{\rho,Q}}$). It follows that $Gspin((A_{\mathbb{E}}^-)_{\rho,Q})$ is indeed $A_{\mathbb{E},\rho,\mathbb{F}Q}^{t^2}$. Now, the proof of Theorem 5.9 shows that the spinor norm map factors through the projection $(g, t) \mapsto t$ before going over to any quotient $(\mathbb{E}^\times/(\mathbb{E}^\times)^2, \text{ or anything else})$. Thus elements of $SO^1((A_{\mathbb{E}}^-)_{\rho,Q})$ are obtained from $g \in A_{\mathbb{E},\rho,\mathbb{F}Q}^{t^2}$ with $t(g) \in (\mathbb{F}^\times)^2$, and the usual interplay with scalars reduces to those g with $t(g) = 1$. But such g preserve Q in the twisted operation, and must lie in $A_{\mathbb{E}}^1$ by the $A_{\mathbb{E},\rho,\mathbb{F}Q}^{t^2}$ condition, which proves the second assertion since the scalars which square to 1 in \mathbb{F} (which again form the kernel) are just ± 1 . This completes the proof of the theorem. \square

The case $d = -1$ of Theorem 7.6 gives back Theorem 5.9 in the following way. We have $\mathbb{E} = \mathbb{F} \times \mathbb{F}$, so that $A_{\mathbb{E}} = A \times A$, and Q lies in the original space (no re-normalization). As $A_{\mathbb{E}}^{(\mathbb{E}^\times)^2} = A^{(\mathbb{F}^\times)^2} \times A^{(\mathbb{F}^\times)^2}$, taking $g \in A^{(\mathbb{F}^\times)^2}$ to be the first coordinate and choosing t such that $t^2 = N_{\mathbb{F}}^A(g)$, we find that the condition $(g, h)Q\overline{(g, h)}^\rho = tQ$ is equivalent to each one of the conditions $gQ\bar{h} = tQ$ and $hQ\bar{g} = tQ$, hence to h being $tQ\bar{g}^{-1}Q^{-1}$ (indeed, with the same reduced norm as g). Hence $A_{\mathbb{E},\rho,\mathbb{F}Q}^{t^2} \cong \tilde{A}^{(\mathbb{F}^\times)^2}$, through the map $(g, t) \mapsto (g, tQ\bar{g}^{-1}Q^{-1})$ (in the opposite direction), so that the second coordinate associated with (g, t) depends on Q , but the isomorphism class of the group does not (see also Proposition 7.7 below for the general case). A similar assertion holds for $A_{\mathbb{E},\rho,Q}^1$.

For the independence property, we observe that apart from the choice of the element Q , there is also the choice of the \mathbb{F} -structure on $A_{\mathbb{E}}$, i.e., of the interpretation of x^ρ for $x \in A_{\mathbb{E}}$. We thus replace this notation by defining σ to be a ring automorphism of order 2 whose restriction to \mathbb{E} is ρ , and which satisfies $(\bar{x})^\sigma = \overline{x^\sigma}$ for every $x \in A$. By replacing σ with another such automorphism, say τ , we get a presentation of $A_{\mathbb{F}}$ as the tensor product of another bi-quaternion algebra over \mathbb{F} with \mathbb{E} . Any element $y \in A_{\mathbb{E}}^\times$ such that $\bar{y}y \in \mathbb{E}^\times$ produces such an automorphism by defining $x^\tau = yx^\sigma y^{-1}$, where the similitude condition is equivalent to the condition $(\bar{x})^\tau = \overline{x^\tau}$ for every $x \in A$: Compare $(\bar{x})^\tau = y\bar{x}^\sigma y^{-1}$ with $\bar{x}^\tau = \bar{y}^{-1}\bar{x}^\sigma\bar{y}$ and use the centrality of $A_{\mathbb{E}}$. The relations we have are contained in the following

Proposition 7.7. (i) Let Q and R be σ -invariant elements of $A_{\mathbb{E}}^-$ (i.e., elements of A^-) such that $|R|^2 \in |Q|^2(\mathbb{F}^\times)^2 N_{\mathbb{F}}^A(A^\times)$. Then the groups $A_{\mathbb{E},\sigma,\mathbb{F}Q}^{t^2}$ and $A_{\mathbb{E},\sigma,\mathbb{F}R}^{t^2}$ as well as $A_{\mathbb{E},\sigma,Q}^1$ and $A_{\mathbb{E},\sigma,R}^1$ are isomorphic. (ii) Assume that $e \in A_{\mathbb{E}}^{\mathbb{F}^\times}$ is such that $ee^{-\sigma}$ is invariant under $x \mapsto \bar{x}^\sigma$, and $S \in A_{\mathbb{E}}^-$ is invariant under $\tau : x \mapsto ee^{-\sigma}x^\sigma e^\sigma e^{-1}$ and satisfies $|S|^2 \in |Q|^2 N_{\mathbb{E}}^{A_{\mathbb{E}}}(e)(\mathbb{F}^\times)^2 N_{\mathbb{F}}^A(A^\times)$. In this case the groups $A_{\mathbb{E},\sigma,\mathbb{F}Q}^{t^2}$ and $A_{\mathbb{E},\tau,\mathbb{F}S}^{t^2}$ and the groups $A_{\mathbb{E},\sigma,Q}^1$ and $A_{\mathbb{E},\tau,R}^1$ are isomorphic. (iii) Let $b \in A_{\mathbb{E}}^\times$ be such that $\bar{b}^\sigma = b$ and $b\bar{b} \in \mathbb{E}^\times$, and define $\eta : x \mapsto bx^\sigma b^{-1}$. If the element $T = Qb^{-1}$ lies in $A_{\mathbb{E}}^-$ then the groups $A_{\mathbb{E},\sigma,\mathbb{F}Q}^{t^2}$ and $A_{\mathbb{E},\eta,\mathbb{F}T}^{t^2}$ coincide, and same assertion holds for $A_{\mathbb{E},\sigma,Q}^1$ and $A_{\mathbb{E},\eta,T}^1$.

Proof. Considering A^- as a space of discriminant 1 again, we find (as in the proof of Proposition 6.5) that we can write $R = rcQ\bar{e} = rcQ\bar{e}^\sigma$ with $r \in \mathbb{F}^\times$ and $c \in A^\times$ (hence $c = c^\sigma$). It follows that conjugation by c takes the group $A_{\mathbb{E},\sigma,\mathbb{F}Q}^{t^2}$ to $A_{\mathbb{E},\sigma,\mathbb{F}R}^{t^2}$ (as conjugation preserves reduced norms, the $N_{\mathbb{E}}^A = t^2$ condition is also preserved), in a manner which commutes with the multiplier maps. This proves part (i), since the groups $A_{\mathbb{E},\sigma,Q}^1$ and $A_{\mathbb{E},\sigma,R}^1$ are defined by the condition $t = 1$ on the larger groups $A_{\mathbb{E},\sigma,\mathbb{F}Q}^{t^2}$ to $A_{\mathbb{E},\sigma,\mathbb{F}R}^{t^2}$. For part (ii), first note that the σ -image of $ee^{-\sigma}$ is its inverse, so that the condition on that element implies that it is a similitude and $(\bar{x})^\tau = \overline{x^\tau}$ holds for every $x \in A_{\mathbb{E}}$. Consider the element $eQ\bar{e}$ of $A_{\mathbb{E}}^-$, whose vector norm is $N_{\mathbb{E}}^{A_{\mathbb{E}}}(e)|Q|^2 \in \mathbb{F}^\times$ by Proposition 5.5 and the assumption on e . The relation between τ and σ and the assumption on Q and e show that this element is τ -invariant, and as the properties of e and τ imply that for $g \in A_{\mathbb{E}}^\times$ we have

$$\overline{ege^{-1}^\tau} = \bar{e}^{-1}\bar{e}^\sigma \cdot \overline{ege^{-1}^\sigma} \cdot \bar{e}^{-\sigma}\bar{e} = \bar{e}^{-1}\bar{g}^\sigma\bar{e},$$

it follows that conjugation by e sends $A_{\mathbb{E},\sigma,\mathbb{F}Q}^{t^2}$ to $A_{\mathbb{E},\tau,\mathbb{F}eQ\bar{e}}^{t^2}$ as well as $A_{\mathbb{E},\sigma,Q}^1$ to $A_{\mathbb{E},\tau,eQ\bar{e}}^1$. As S is related to $eQ\bar{e}$ in the same manner as Q and R , part (ii) now follows from part (i), using conjugation by an appropriate τ -invariant element $d \in A_{\mathbb{E}}^\times$. Now, the fact that $Q \in A^-$ and the element T from part (iii) lies in $A_{\mathbb{E}}^-$ imply $T^\eta = -\bar{T}^\eta = -b\overline{Qb^{-1}^\sigma}b^{-1} = +b\bar{b}^{-\sigma}Qb^{-1}$, and this gives $Qb^{-1} = T$ again by our assumption on b . Furthermore, for $g \in A$ we have $gT\bar{g}^\eta = gQ\bar{g}^\sigma b^{-1}$ by the definitions of T and η , so that the $A_{\mathbb{E},\sigma,\mathbb{F}Q}^{t^2}$ and $A_{\mathbb{E},\eta,\mathbb{F}S}^{t^2}$ conditions coincide, with the same multiplier t . This proves part (iii) (for both the Gspin and spin groups), and completes the proof of the proposition. \square

Note that part (i) of Proposition 7.7 is a special case of part (ii) there, but it is required for the proof of the latter part. We also remark that for τ and σ which are connected as in part (ii) of Proposition 7.7, the subring which τ stabilizes is isomorphic to A . Indeed, if $x = x^\tau = aa^{-\sigma}x^\sigma a^\sigma a^{-1}$ then $a^{-1}xa$ is σ -invariant, so that conjugation by a maps A onto this subring (in fact, the Skolem–Noether Theorem implies that every automorphism on $A_{\mathbb{E}}$ which stabilizes a subring which is isomorphic to A must arise in this way). This is related to the fact that the vector norm relation is based on $N_{\mathbb{F}}^A(A^\times)$ also when τ is involved. On the other hand, η from part (iii) of that proposition may arise from a non-isomorphic bi-quaternion algebra over \mathbb{F} . E.g., assuming that $Q \in C_0$, any element $b \in \mathbb{F} \oplus \mathbb{E}_0 C_0$ such that $Tr_{\mathbb{E}}^{C_{\mathbb{E}}}(Qb^{-1}) = 0$ satisfies all the assumptions (including $Qb^{-1} \in A_{\mathbb{E}}^-$), and we have seen in the paragraph preceding Corollary 4.6 that up to letting $Q \in C_0$ vary, this covers all the bi-quaternion algebras which arise as the tensor product of B with some quaternion algebra over \mathbb{F} which becomes $C_{\mathbb{E}}$ over \mathbb{E} . We may also apply this construction with $b \in B_0 \otimes Q$, yielding an operation on both B in C . We remark that by Proposition 2.18 of [KMRT], any two involutions $x \mapsto \bar{x}^\sigma$ and $x \mapsto \bar{x}^\tau$ must be related through conjugation by some element b satisfying $\bar{b}^\sigma = b$, with the similitude condition to preserve commutativity with $\iota_B \otimes \iota_C \otimes Id_{\mathbb{E}}$. It seems

likely that the similitude condition implies the existence of $Q \in A^- \cap A^\times$ such that $Qb^{-1} \in A_{\mathbb{E}}^-$, but we have not checked this in detail. Was this the case, Proposition 7.7 would relate all the involutions which commute with $\iota_B \otimes \iota_C \otimes Id_{\mathbb{E}}$ to one another, yielding an \mathbb{F} -structure invariance result.

Fixing Q and σ back again (and writing ρ for the automorphism of $A_{\mathbb{E}}$ as before), the assertion involving isotropy in this case is

Corollary 7.8. *If the space becomes isotropic when one extends scalars to \mathbb{E} then the $G\text{spin}$ group is a “quaternionic GSU group”, consisting of the matrices $g \in M_2(B_{\mathbb{E}})$, for some quaternion algebra $B_{\mathbb{E}}$ over \mathbb{E} which comes from a quaternion algebra B over \mathbb{F} , which multiply a diagonal matrix of the sort $\begin{pmatrix} -\delta & 0 \\ 0 & 1 \end{pmatrix} \in M_2(\mathbb{F})$ with δ determined up to multiplication from $N_{\mathbb{F}}^B(B^\times)$, through the action $g : X \mapsto gX\bar{g}^t$. The GSU condition means that the reduced norm of these matrices equals the square of the multiplier of this element. The spin group is the associated “quaternionic special unitary group”, of matrices which preserve this element and have reduced norm 1. In case the space splits two hyperbolic planes over \mathbb{E} , the $G\text{spin}$ group becomes the $GSU_{\mathbb{E},\rho}$ group of a unitary space of dimension 4 and determinant (or discriminant) 1 over \mathbb{E} (with ρ), and the spin group is the associated special unitary group.*

Proof. In the first case $A_{\mathbb{E}}$ is isomorphic to $M_2(B_{\mathbb{E}})$ for some quaternion algebra B over \mathbb{F} . We may normalize (using parts (ii) and (iii) of Proposition 7.7 if necessary) the involutions such that $A = M_2(B)$, i.e., with $C = M_2(\mathbb{F})$, and as every element of \mathbb{F}^\times is a norm from C_0 , part (i) of Proposition 7.7 allows us to restrict attention to spaces $(M_2(B_{\mathbb{E}}))_{\rho,Q}^-$ with $Q \in M_2(\mathbb{F})_0$. Moreover, as in the proof of Corollary 6.7, we may take Q of the form $\begin{pmatrix} 0 & \delta \\ 1 & 0 \end{pmatrix}$, and part (i) of Proposition 7.7 and Lemma 6.6 allows us to take δ from a set of representatives for $\mathbb{F}^\times / N_{\mathbb{F}}^B(B^\times)$. By Lemma 1.3, the condition $g \in A_{\mathbb{E},\sigma,\mathbb{F}Q}^{t^2}$ (or $g \in GL_2^{t^2}(B_{\mathbb{E}})_{\sigma,\mathbb{F}Q}$) becomes $g \begin{pmatrix} -\delta & 0 \\ 0 & 1 \end{pmatrix} \iota_B(g)^{t\rho} = t \begin{pmatrix} -\delta & 0 \\ 0 & 1 \end{pmatrix}$ for some $t = t(g) \in \mathbb{F}^\times$ and $N_{\mathbb{E}}^{M_2(B_{\mathbb{E}})}(g) = t(g)^2$, while elements of $A_{\mathbb{E},\sigma,Q}^1 = GL_2^1(B_{\mathbb{E}})_{\sigma,Q}$ have reduced norm 1 and their action leaves the element $\begin{pmatrix} -\delta & 0 \\ 0 & 1 \end{pmatrix}$ invariant. This proves the first assertion.

If after tensoring $(A_{\mathbb{E}}^-)_{\rho,Q}$ with \mathbb{E} we get a 2-dimensional isotropic subspace (and then we even get a 3-dimensional such space) then $B_{\mathbb{E}}$ also splits, hence B may be presented as $(\frac{-d,\varepsilon}{\mathbb{F}})$ for some $\varepsilon \in \mathbb{F}^\times$ (which is unique up to $(\mathbb{F}^\times)^2$ and multiplication by $+d$). The model we thus take for B is the algebra $(\mathbb{E}, \rho, \varepsilon)$, and by Lemma 1.2, $M \mapsto M^\rho$ is $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto \begin{pmatrix} 0 & \varepsilon \\ 1 & 0 \end{pmatrix} \begin{pmatrix} a^\rho & b^\rho \\ c^\rho & d^\rho \end{pmatrix} \begin{pmatrix} 0 & \varepsilon \\ 1 & 0 \end{pmatrix} / \varepsilon$ on $B_{\mathbb{E}} = M_2(\mathbb{E})$. Thus the operation $g \mapsto \bar{g}^\rho$ or $g \mapsto \iota_B(g)^{t\rho}$ involves the map $M \mapsto \begin{pmatrix} 0 & \varepsilon \\ 1 & 0 \end{pmatrix} M^\rho \begin{pmatrix} 0 & \varepsilon \\ 1 & 0 \end{pmatrix} / \varepsilon$ on the entries (in addition to adjoint or transposition of 2×2 matrices over $B_{\mathbb{E}}$). It follows that each entry from $B_{\mathbb{E}}$ in $\begin{pmatrix} 0 & \delta \\ 1 & 0 \end{pmatrix}$ or $\begin{pmatrix} -\delta & 0 \\ 0 & 1 \end{pmatrix}$ appearing in the definition of the groups (which we write as $\begin{pmatrix} 0 & \delta I \\ I & 0 \end{pmatrix}$ or $\begin{pmatrix} -\delta I & 0 \\ 0 & I \end{pmatrix}$ since $B_{\mathbb{E}}$ is the matrix algebra $M_2(\mathbb{E})$) is replaced by the corresponding multiple of $\begin{pmatrix} 0 & \varepsilon \\ 1 & 0 \end{pmatrix}$ when we apply ρ on the \mathbb{E} -entries of matrices in $M_2(B_{\mathbb{E}}) = M_4(\mathbb{E})$. Applying Lemma 1.3 to the 2×2 -entries changes each such $\begin{pmatrix} 0 & \varepsilon \\ 1 & 0 \end{pmatrix}$ to $\begin{pmatrix} -\varepsilon & 0 \\ 0 & 1 \end{pmatrix}$. Therefore, the $A_{\mathbb{E},\sigma,\mathbb{F}Q}^{t^2} = GL_2^{t^2}(B_{\mathbb{E}})_{\sigma,\mathbb{F}Q}$ condition just means

being in the $GSU_{\mathbb{E},\rho}$ group of a 4-dimensional space over \mathbb{E} with a sesqui-linear form having an orthogonal basis with norms $\delta\varepsilon$, $-\varepsilon$, $-\delta$, and 1 (hence the determinant is 1 modulo $N_{\mathbb{F}}^{\mathbb{E}}(\mathbb{E}^{\times})$), while the group $A_{\mathbb{E},\sigma,Q}^1 = GL_2^1(B_{\mathbb{E}})_{\sigma,Q}$ is the corresponding $SU_{\mathbb{E},\rho}$ group. This proves the corollary. \square

We emphasize that only unitary spaces with determinant (or discriminant) 1 appear in this context.

In this isotropic case we have the following equivalent representations:

Corollary 7.9. *The groups $GL_2^{t^2}(B_{\mathbb{E}})_{\sigma,\mathbb{F}Q}$ and $GL_2^1(B_{\mathbb{E}})_{\sigma,Q}$ with $Q = \begin{pmatrix} 0 & \delta \\ 1 & 0 \end{pmatrix}$ are the Gspin and spin groups of the space $\mathbb{E} \oplus B$, embedded as $(y, \lambda) \mapsto \begin{pmatrix} \lambda & -\delta y \\ y^{\rho} & -\lambda \end{pmatrix}$ or as $(y, \lambda) \mapsto \begin{pmatrix} \delta y & \lambda \\ \lambda & y^{\rho} \end{pmatrix}$, with the vector norm being $N_{\mathbb{F}}^B(\lambda) - \delta N_{\mathbb{F}}^{\mathbb{E}}(y)$ and the actions being $g : M \mapsto gM\bar{g}$ and $g : X \mapsto gX\iota_B(g)^t$ respectively. The $GSU_{\mathbb{E},\rho}$ and $SU_{\mathbb{E},\rho}$ groups of a 4-dimensional unitary space of discriminant 1 are the Gspin and spin groups of the direct sum of 3 copies of \mathbb{E} , with the vector norm of (z, w, y) being $N_{\mathbb{F}}^{\mathbb{E}}(z) - \varepsilon N_{\mathbb{F}}^{\mathbb{E}}(w) - \delta N_{\mathbb{F}}^{\mathbb{E}}(y)$. It may be embedded in $M_4(\mathbb{E})$ by replacing λ by $\begin{pmatrix} z & \varepsilon w \\ w^{\rho} & z^{\rho} \end{pmatrix}$ with z and w from \mathbb{E} in the two representations from above. In addition, these groups act via the operation $g : T \mapsto gTg^t$ on the subspace of $M_4^{as}(\mathbb{E})$ consisting of those matrices $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ in which the anti-symmetric matrices a and $d = \begin{pmatrix} 0 & -y \\ y & 0 \end{pmatrix}$ satisfies $a = \delta d^{\rho}$ while $b = -c^t$ takes the form $\begin{pmatrix} \varepsilon w & -z \\ z^{\rho} & -w^{\rho} \end{pmatrix}$, as well as through $\begin{pmatrix} a & b \\ c & d \end{pmatrix} : S \mapsto \begin{pmatrix} a & b \\ c & d \end{pmatrix} S \begin{pmatrix} d^t & -b^t \\ -c^t & a^t \end{pmatrix}$ on another embedding of the three copies of \mathbb{E} into the complement of $\mathfrak{sp}_4(\mathbb{E})$ in $M_4(\mathbb{E})$.*

Proof. Recall that $(A_{\mathbb{E}}^-)_{\rho,Q}$ is $(Q^{\perp} \subseteq A^-) \oplus \mathbb{E}_0 Q$, and the off-diagonal matrices which lie in Q^{\perp} are spanned by $\begin{pmatrix} 0 & -\delta \\ 1 & 0 \end{pmatrix}$. Adding $\mathbb{E}_0 Q$ yields the first representation, on which the action is via $g : M \mapsto \frac{gM\bar{g}}{t(g)}$, and the second one, on which the groups operated via $g : X \mapsto \frac{gX\iota_B(g)^t}{t(g)}$, arises from Lemma 1.3 as in the proofs of Corollaries 5.11 and 6.8 (with the norm being some generalization of the Moore determinant). The two representations of the $GSU_{\mathbb{E},\rho}$ and $SU_{\mathbb{E},\rho}$ groups are also obtained by applying Lemma 1.3 to the entries as 2×2 matrices, as the proofs of Corollaries 5.11 and 6.8 do, while recalling that B is embedded into $M_2(\mathbb{E})$ as $(\mathbb{E}, \rho, \varepsilon)$. This proves the corollary. \square

We remark that for the case where $A = M_4(\mathbb{F})$ we have used a different choice of Q in Corollaries 6.7 and 6.8 in order to obtain the classical symplectic group. Choosing this Q for Corollaries 7.8 and 7.9 yields the $GSU_{\mathbb{E},\rho}$ and $SU_{\mathbb{E},\rho}$ conditions for an anti-diagonal symmetric matrix (which is explicitly $\begin{pmatrix} 0 & E \\ E & 0 \end{pmatrix}$ with $E = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$). On the other hand, here the splitting of $A_{\mathbb{E}}$ might come from $A = M_2(B)$ with B which is non-split over \mathbb{F} , so that we have many groups in this case. In any case, we restrict attention to the classical unitary groups (of diagonal matrices), as any unitary group is conjugate to a classical one.

8 Relations with the Exterior Square

For the groups arising from bi-quaternion algebras, hence related to 4×4 matrices, there are representations which are equivalent to those presented here. Given a field \mathbb{M} , the group $GL_4(\mathbb{M})$ operates on the 6-dimensional exterior square $\bigwedge^2 \mathbb{M}^4$ of the natural representation space \mathbb{M}^4 , and if we denote the canonical basis for \mathbb{M}^4 by e_i , $1 \leq i \leq 4$ for \mathbb{M}^4 then the 6 elements $e_i \wedge e_j$ with $1 \leq i < j \leq 4$ form a basis for $\bigwedge^2 \mathbb{M}^4$. The map taking u and v from $\bigwedge^2 \mathbb{M}^4$ to the multiple of $e_1 \wedge e_2 \wedge e_3 \wedge e_4$ which equals $u \wedge v \in \bigwedge^4 \mathbb{M}^4$ is bilinear and symmetric (hence we denote this value by $\langle u, w \rangle$), and in fact independent, up to rescaling, of the choice of basis. The connection which we have arises from the following

Lemma 8.1. *For any n , the natural representation of $GL_n(\mathbb{M})$ on $\bigwedge^2 \mathbb{M}^n$ is equivalent to its representation on the space $M_n^{as}(\mathbb{M})$ of anti-symmetric matrices via $g : T \mapsto gTg^t$. For $n = 4$ the equivalence preserves the bilinear forms, where on $\bigwedge^2 \mathbb{M}^4$ we take the bilinear form defined above and on $M_n^{as}(\mathbb{M})$ we use the Pfaffian.*

Proof. Using the standard basis for \mathbb{M}^n , consider the map which takes, for all $1 \leq i < j \leq n$, the basis element $e_i \wedge e_j$ of $\bigwedge^2 \mathbb{M}^n$ to the anti-symmetric matrix $E_{ij} - E_{ji}$. The fact that for $g = (a_{kl})$ we have $(g(E_{ij} - E_{ji})g^t)_{kl} = a_{ki}a_{lj} - a_{kj}a_{li}$ shows that this is indeed an equivalence of representations. For $n = 4$ we find that $e_1 \wedge e_2$ and $e_3 \wedge e_4$ as well as $e_1 \wedge e_4$ and $e_2 \wedge e_3$ span hyperbolic planes, while $e_1 \wedge e_3$ and $e_2 \wedge e_4$ span another hyperbolic plane but with the sign inverted. As this is in correspondence with the values bilinear form arising from the Pfaffian takes on the images of these vectors in $M_n^{as}(\mathbb{M})$, this completes the proof of the lemma. \square

Given $g \in GL_4(\mathbb{M})$ and u and v from $\bigwedge^2 \mathbb{M}^4$, we have $\langle gu, gv \rangle = \det g \langle u, v \rangle$. This is so, since $gu \wedge gv$ is $\langle u, v \rangle$ times the image of the generator $e_1 \wedge e_2 \wedge e_3 \wedge e_4$ of $\bigwedge^4 \mathbb{M}^4$ via the \bigwedge^4 -action of g , and the latter action is multiplication by the determinant by definition. Lemma 8.1 and the isomorphism of representations appearing in the proof of Corollary 5.11 thus provide an alternative proof for Proposition 5.5, via appropriate restriction of scalars.

Assume now that $ch\mathbb{M} \neq 2$. Then sums and differences of complementary pairs form an orthogonal basis for \mathbb{M}^4 . All the representations in this Section will be variants of this one, using this basis. The first one is

Proposition 8.2. *$SL_4(\mathbb{F})$ is the spin group of the quadratic space $\bigwedge^2 \mathbb{M}^4$, and the Gspin group is the double cover $\widetilde{GL}_4^{(\mathbb{F}^\times)^2}(\mathbb{F})$, operating with division by the chosen square root. The classical $Sp_4(\mathbb{F})$ and $GSp_4(\mathbb{F})$ are the spin and Gspin groups of the subspace of $\bigwedge^2 \mathbb{M}^4$ which is spanned by $e_1 \wedge e_2$, $e_1 \wedge e_4$, $e_2 \wedge e_3$, $e_3 \wedge e_4$, and $e_1 \wedge e_3 - e_2 \wedge e_4$.*

Proof. The first two assertions follow directly from Corollary 5.11 by taking $\mathbb{M} = \mathbb{F}$ in Lemma 8.1. For the last two assertions we apply Corollary 6.8,

observing that our element Q is taken through all these maps to $e_1 \wedge e_3 + e_2 \wedge e_4$, the orthogonal complement of which is spanned by the asserted vectors. This proves the proposition. \square

When we consider spaces of general discriminant d , let \mathbb{E} be $\mathbb{F}(\sqrt{d})$ with Galois automorphism ρ . Considering the bi-quaternion algebra $A = M_2(B)$ with $B = (\frac{d, \varepsilon}{\mathbb{F}})$ (which splits over \mathbb{E}), and choosing some $\delta \in \mathbb{F}^\times$, we find that

Proposition 8.3. *The $SU_{\mathbb{E}, \rho}$ group of a space with an orthogonal basis having norms $\delta\varepsilon$, $-\varepsilon$, $-\delta$, and 1 is the spin group of the \mathbb{F} -subspace of $\bigwedge^2 \mathbb{E}^4$ which is the direct sum of $\mathbb{F}(e_1 \wedge e_4 - e_2 \wedge e_3)$, $\mathbb{E}_0(e_1 \wedge e_4 + e_2 \wedge e_3)$, $\mathbb{F}(e_2 \wedge e_4 - \varepsilon e_1 \wedge e_3)$, $\mathbb{E}_0(e_2 \wedge e_4 + \varepsilon e_1 \wedge e_3)$, $\mathbb{F}(e_3 \wedge e_4 + \delta e_1 \wedge e_2)$, and $\mathbb{E}_0(e_3 \wedge e_4 - \delta e_1 \wedge e_2)$. The Gspin group is the associated $GSU_{\mathbb{E}, \rho}$ group.*

Proof. Multiplying each entry of the elements of the second representation appearing in Corollary 7.8 by $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ from the right takes, if $\lambda \in B = (\frac{d, \varepsilon}{\mathbb{F}})$, this incarnation of $\mathbb{E} \oplus B$ to the asserted direct sum (the first two generate the image of the diagonal matrices in B , the second two yield those of the off-diagonal matrices, and the latter two give the image of \mathbb{E}). The assertion now follows from Corollary 7.9 and Lemma 8.1 for $\mathbb{M} = \mathbb{E}$. This proves the proposition. \square

Note that unlike in Proposition 8.2, here the image of Q became the element $e_3 \wedge e_4 - \delta e_1 \wedge e_2$, appearing in the 6th direct summand in Proposition 8.3.

We now consider the case where A^- is isotropic, but does not necessarily split more than one hyperbolic plane. Then A is isomorphic to $M_2(B)$ for some quaternion algebra B , and by taking \mathbb{K} to be a quadratic extension of \mathbb{F} , whose Galois automorphism we denote η , which splits B , we may write $B \cong (\mathbb{K}, \eta, \varepsilon)$ for some $\varepsilon \in \mathbb{F}^\times$. In this case we have

Proposition 8.4. *The groups $\widetilde{GL}_2^{(\mathbb{F}^\times)^2}(B)$ and $GL_2^1(B)$ are the Gspin and spin groups of the \mathbb{F} -subspace of $\bigwedge^2 \mathbb{K}^4$ which is obtained as the direct sum of the four spaces $\mathbb{F}(e_1 \wedge e_4 - e_2 \wedge e_3)$, $\mathbb{K}_0(e_1 \wedge e_4 + e_2 \wedge e_3)$, $\mathbb{F}(e_2 \wedge e_4 - \varepsilon e_1 \wedge e_3)$, $\mathbb{K}_0(e_2 \wedge e_4 + \varepsilon e_1 \wedge e_3)$, and the hyperbolic plane $\mathbb{F}e_1 \wedge e_2 \oplus \mathbb{F}e_3 \wedge e_4$. The quaternionic symplectic groups $Sp_B\begin{pmatrix} -\delta & 0 \\ 0 & 1 \end{pmatrix}$ and $GSp_B\begin{pmatrix} -\delta & 0 \\ 0 & 1 \end{pmatrix}$ are the spin and Gspin groups of the subspace which is the direct sum of the first four 1-dimensional spaces with $\mathbb{F}(e_3 \wedge e_4 + \delta e_1 \wedge e_2)$. If d is a non-trivial discriminant and $\mathbb{E} = \mathbb{F}(\sqrt{d})$ has Galois automorphism ρ over \mathbb{F} , then $GL_2^{t^2}(B_{\mathbb{E}})_{\rho, \mathbb{F}Q}$ and $GL_2^1(B_{\mathbb{E}})_{\rho, Q}$ are the Gspin and spin groups of the direct sum of the latter 5-dimensional space and $\mathbb{E}_0(e_3 \wedge e_4 - \delta e_1 \wedge e_2)$.*

Proof. The first space is the image of $M_2(B)^-$ under the isomorphisms from Corollary 5.11 and Lemma 8.1 with $\mathbb{M} = \mathbb{K}$. For the second and third space we note that our vector Q is the same matrix from Proposition 8.3, hence has the same image. This completes the proof of the proposition as we did for Propositions 8.2 and 8.3, where for the latter assertion we need to apply Lemma 8.1 with $\mathbb{M} = \mathbb{K}\mathbb{E}$. \square

We remark that if B is not split but \mathbb{E} splits B then taking $\mathbb{K} = \mathbb{E}$ in Proposition 8.4 yields Proposition 8.3 again.

In the general case, where A might be a division algebra, we write $A = B \otimes C$ take \mathbb{L} to be a quadratic extension of \mathbb{F} (with Galois automorphism ω) which splits C . As we have seen that our choice of Q may be taken from C_0 , we assume that $C \cong (\mathbb{L}, \omega, \delta)$ with the same δ . Hence in general we have

Proposition 8.5. *A space for which the groups $\tilde{A}^{(\mathbb{F}^\times)^2}$ and A^1 appear as the Gspin and spin groups is the direct sum of the \mathbb{F} -spaces $\mathbb{L}_0(e_1 \wedge e_4 - e_2 \wedge e_3)$, $\mathbb{K}_0(e_1 \wedge e_4 + e_2 \wedge e_3)$, $\mathbb{F}(e_2 \wedge e_4 - \varepsilon e_1 \wedge e_3)$, $\mathbb{K}_0(e_2 \wedge e_4 + \varepsilon e_1 \wedge e_3)$, $\mathbb{L}_0(e_3 \wedge e_4 + \delta e_1 \wedge e_2)$, and $\mathbb{F}(e_3 \wedge e_4 - \delta e_1 \wedge e_2)$ inside $\bigwedge^2(\mathbb{KL})^4$. The stabilizers $A_{\mathbb{F}Q}^\times$ and A_Q^\times are the Gspin and spin groups of the direct sum of the first 5 spaces, if $Q \in A^-$ is chosen to be $\begin{pmatrix} 0 & \delta \\ 1 & 0 \end{pmatrix} \in C_0$. With this choice of Q , a space of discriminant d for which $A_{\mathbb{E}, \rho, \mathbb{F}Q}^{t_2}$ is the Gspin group and $A_{\mathbb{E}, \rho, Q}^1$ is the spin group, where $\mathbb{E} = \mathbb{F}(\sqrt{d})$ and ρ is its Galois automorphism, is obtained by adding $\mathbb{E}_0(e_3 \wedge e_4 - \delta e_1 \wedge e_2)$ to the latter 5-dimensional space.*

Proof. The embedding of C inside $M_2(\mathbb{L})$ embeds A into $M_2(B_{\mathbb{L}})$, hence A^- into $M_2(B_{\mathbb{L}})^-$. As the two scalar entries represent scalar entries of $C \subseteq M_2(\mathbb{L})$, they are related to one another via ω and multiplication by δ , yielding the two latter spaces under the maps from Corollary 5.11 and Lemma 8.1 with $\mathbb{M} = \mathbb{KL}$. As the remaining entries come from $\lambda \in B_{\mathbb{L}}$ which in fact lies in $B_0 \oplus \mathbb{L}_0$, the off-diagonal entries of $\lambda \in M_2(\mathbb{KL})$ are related through η and multiplication by ε , yielding the middle two spaces via these identifications. The diagonal entries must therefore be negated by $\eta\omega$, hence come from $\mathbb{K}_0 \oplus \mathbb{L}_0$, which gives the first two subspaces. This proves the assertions about $\tilde{A}^{(\mathbb{F}^\times)^2}$ and A^1 , and the ones about the symplectic groups follow since we use the same element $Q = \begin{pmatrix} 0 & \delta \\ 1 & 0 \end{pmatrix}$ as in Propositions 8.3 and 8.4 (note that this element belongs to C as a subalgebra of $M_2(\mathbb{L})$ in our normalizations). For the remaining assertions we use again the same element Q , and we argue as in Propositions 8.3 and 8.4, where \mathbb{M} is now taken to be $\mathbb{KL}\mathbb{E}$ in Lemma 8.1. This proves the proposition. \square

We remark again that our choice of Q in Proposition 8.5, which looks rather special, is entirely general when we normalize B and C appropriately. Note that taking $\mathbb{K} = \mathbb{L}$ in Proposition 8.5 in case A is not a division algebra, or $\mathbb{L} = \mathbb{E}$ in case A is a division algebra but $A_{\mathbb{E}}$ is not, does not give us Proposition 8.4 again. This is so, since the split algebra C is normalized as $M_2(\mathbb{F})$ in Proposition 8.4, but as some subalgebra $(\mathbb{L}, \omega, \delta)$ of $M_2(\mathbb{L})$, with the \mathbb{F} -structure from Lemma 1.2, in Proposition 8.5.

9 Dimension 8, Isotropic, Discriminant 1

The spaces we consider here are described in the following

Lemma 9.1. *The direct sum of a hyperbolic plane with an Albert form arising from a presentation of a bi-quaternion algebra A as the tensor product $B \otimes C$ of*

two quaternion algebras over \mathbb{F} is isotropic of dimension 8 and discriminant 1. Any isotropic 8-dimensional space of discriminant 1 is obtained, up to rescaling and isometries, in this way.

Proof. Recall that a space is isotropic if and only if it has a hyperbolic plane as a direct summand, and that a hyperbolic plane is isometric to its rescalings. The lemma now follows directly from Lemma 5.1. \square

We thus fix a bi-quaternion algebra $A = B \otimes C$ with the (orthogonal) involution $\iota_B \otimes \iota_C : x \mapsto \bar{x}$ as above, and the space from Lemma 9.1 may be denoted $A^- \oplus H$ (where H stands for a hyperbolic plane). It will be useful to embed this space into $M_2(A)$ by taking the sum of $u \in A^-$, $-p$ times one generator of the hyperbolic plane, and q times the second generator, to the matrix $U = \begin{pmatrix} u & -p \\ q & -\bar{u} \end{pmatrix}$. We shall henceforth identify $A^- \oplus H$ with the space of those matrices. The norm $|U|^2 = u\bar{u} - pq$ (see Lemma 5.2) resembles the “Moore-like determinant” of $M_2(B)^-$ from the proof of Corollary 5.4, but now with a subset of a bi-quaternion algebra rather than a usual quaternion algebra.

Let $\widehat{GSp}_A \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ be the group of (invertible) matrices $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M_2(A)$ which preserve the space $\mathbb{F} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ under the operation $\begin{pmatrix} a & b \\ c & d \end{pmatrix} : M \mapsto \begin{pmatrix} a & b \\ c & d \end{pmatrix} M \begin{pmatrix} \bar{d} & -\bar{b} \\ -\bar{c} & \bar{a} \end{pmatrix}$ on $M_2(A)$. An element $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M_2(A)$ lies in $\widehat{GSp}_A \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ if and only if $a\bar{b}$ and $c\bar{d}$ lie in A^- and there exists an element $m \in \mathbb{F}^\times$ such that $a\bar{d} + b\bar{c} = m$ (and equivalently $d\bar{a} + c\bar{b} = m$). The fact that our matrix $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ equals its inverse implies that if $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \widehat{GSp}_A \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ then so is $\begin{pmatrix} \bar{d} & -\bar{b} \\ -\bar{c} & \bar{a} \end{pmatrix}$ (with the same multiplier m), so that $\bar{c}a$ and $\bar{d}b$ belong to A^- and $\bar{a}d + \bar{c}b = m$ as well as $\bar{d}a + \bar{c}b = m$. We call these relations *the GSp relations*, and the map taking an element of $\widehat{GSp}_A \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ to the scalar m by which its action multiplies $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ is a group homomorphism $\widehat{GSp}_A \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \rightarrow \mathbb{F}^\times$. Now, any element $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ of $\widehat{GSp}_A \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, with multiplier m , satisfies $N_{\mathbb{F}}^{M_2(A)} \begin{pmatrix} a & b \\ c & d \end{pmatrix}^2 = m^8$ (for the reduced norm of the degree 8 algebra $M_2(A)$ over \mathbb{F}). The function $\frac{N_{\mathbb{F}}^{M_2(A)}}{m^4}$ is thus a group homomorphism $\widehat{GSp}_A \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \rightarrow \{\pm 1\}$, and we define $GSp_A \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ to be its kernel. We shall see in Lemma 9.3 below that unless $A = M_4(\mathbb{F})$, the latter homomorphism is trivial and $\widehat{GSp}_A \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ and $GSp_A \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ coincide, so that the reduced norm condition becomes redundant. The kernel of the restriction of m to $\widehat{GSp}_A \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ is just the symplectic group $Sp_A \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ consisting of those matrices which preserve the element $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ itself under this operation, and have reduced norm 1.

We begin our analysis of this group with the following

Lemma 9.2. *Let $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ be an element of $\widehat{GSp}_A \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, with multiplier m . If $a \in A^\times$ then $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is the product $\begin{pmatrix} 1 & 0 \\ \beta & 1 \end{pmatrix} \begin{pmatrix} a & 0 \\ 0 & m\bar{a}^{-1} \end{pmatrix} \begin{pmatrix} 1 & \alpha \\ 0 & 1 \end{pmatrix}$ with α and β from A^- . Moreover, these matrices lie in $GSp_A \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$.*

Proof. $b\bar{a}$ and $\bar{a}c$ lie in A^- by the GSp relations, and since $a \in A^\times$ we find that $\alpha = a^{-1}b = a^{-1}(b\bar{a})\bar{a}^{-1}$ and $\beta = ca^{-1} = \bar{a}^{-1}(\bar{a}c)a^{-1}$ are also in A^- . Hence

$b = a\alpha$ and $c = \beta a$, and as $d\bar{a} + c\bar{b} = m$ by the GSp relations, it follows that $d = m\bar{a}^{-1} + \beta a\alpha$. But this is easily seen to be the value of the asserted product. We claim that the multipliers lie in $GSp_A\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, which will prove the second assertion. In order to evaluate the reduced norms we may extend scalars to a splitting field of A , and then we evaluate 8×8 determinants. Now, the unipotent elements become unipotent 8×8 matrices, hence they have determinant 1, in correspondence with their multipliers being 1. The diagonal matrix is a block matrix, hence has reduced norm $N_{\mathbb{F}}^A(a) \cdot N_{m\mathbb{F}}^A(\bar{a}^{-1}) = m^4$, which proves the claim as the multiplier of this element is m . This completes the proof of the lemma. \square

The following technical result will be very useful in what follows.

Lemma 9.3. *Given any element $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GSp_A\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, there exists $v \in A^-$ such that the result of multiplying it from the left by a matrix of the sort $\begin{pmatrix} 1 & v \\ 0 & 1 \end{pmatrix}$ has an invertible upper left entry. In fact, this is the case for “almost any $v \in A^-$ ”. In addition, $GSp_A\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ has index 2 in $\widehat{GSp}_A\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ if and only if A is split.*

Proof. The upper left entry of the product in question is $a + vc$. By fixing an arbitrary non-zero $w \in A^-$, consider the expression $N_{\mathbb{F}}^A(a + swc)$ as a function of $s \in \mathbb{F}$. It is a polynomial of degree not exceeding 4 in s . Now, if $a \in A^\times$ then this polynomial never vanishes at $s = 0$. Hence for all but at most 4 (non-zero) values of s (the roots of this polynomial), $v = sw$ has the desired property (in fact, this polynomial has at most two roots, as it decomposes as the product of the global constant $N_{\mathbb{F}}^A(a)$ with the polynomial $N_{\mathbb{F}}^A(1 + swca^{-1})$ and the latter expression is a square by the proof of Lemma 9.2 and Lemma 9.5 below). In case $a = 0$ we know that c must be invertible (for the GSp relation $a\bar{d} + b\bar{c} = mI$ to be possible), and then with any anisotropic v the element $a + vc = vc$ is invertible. This covers the case where A is a division algebra (since then either $a = 0$ or $a \in A^\times$), so assume that $A = M_2(B)$ for B a quaternion algebra over \mathbb{F} . We only have to consider the case where a is non-zero singular 2×2 matrices over B . Hence there exist σ and τ in B , not both zero, such that $a\begin{pmatrix} \sigma \\ \tau \end{pmatrix} = 0$ as a 2-vector over B . As $a \neq 0$, this allows us to construct some $y \in GL_2(B)$ such that ay has right column 0. It is then easy to find $x \in GL_2(B)$ such that $xay = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$, and by multiplying $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ from the left by $\begin{pmatrix} y & 0 \\ 0 & \bar{y}^{-1} \end{pmatrix}$ and from the right by $\begin{pmatrix} x & 0 \\ 0 & \bar{x}^{-1} \end{pmatrix}$ we may consider only elements with $a = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$. Indeed, the right multiplication by $\begin{pmatrix} y & 0 \\ 0 & \bar{y}^{-1} \end{pmatrix}$ does not affect our assertions, and conjugating $\begin{pmatrix} 1 & v \\ 0 & 1 \end{pmatrix}$ by $\begin{pmatrix} x & 0 \\ 0 & \bar{x}^{-1} \end{pmatrix}$ yields just $\begin{pmatrix} 1 & xv\bar{x} \\ 0 & 1 \end{pmatrix}$, with $xv\bar{x} \in A^-$. Write c as $\begin{pmatrix} \lambda & \mu \\ \kappa & \nu \end{pmatrix}$, with entries from B . The GSp condition $\bar{c}a \in A^-$ implies $\nu = 0$ and $\kappa \in \mathbb{F}$, and as $\bar{d}a + \bar{b}c = mI$ is invertible, we find that $\mu \in B^\times$. Choose w of the sort $\begin{pmatrix} \sigma & -r \\ 1 & -\bar{\sigma} \end{pmatrix}$, and consider the polynomial in s defined above. As $a + swc = \begin{pmatrix} 1+s\sigma\lambda+s\tau\kappa & s\sigma\mu \\ s\lambda-s\kappa\bar{\sigma} & s\mu \end{pmatrix}$, one may use Lemma 2.1 to evaluate the coefficients of the powers of s in the resulting expression from the formula from Proposition 5.3, which gives us $s^2 N_{\mathbb{F}}^B(\mu) + 2s^3 \kappa N_{\mathbb{F}}^B(\mu)|w|^2 + s^4 N_{\mathbb{F}}^A(w)N_{\mathbb{F}}^A(c)$. Evaluating $N_{\mathbb{F}}^A(c)$ as $\kappa^2 N_{\mathbb{F}}^B(\mu)$ (by Proposition 5.3) and using Corollary 5.4,

this polynomial is (up to the global scalar $N_{\mathbb{F}}^B(\mu) \in \mathbb{F}^\times$) just $s^2(1 + \kappa|w|^2s)^2$. Hence our upper entry is invertible for all non-zero s if w is isotropic or c is not invertible (i.e., $\kappa = 0$), and we also have to omit the value $s = -\frac{1}{\kappa|w|^2}$ otherwise. As these are at most two values of s and $ch\mathbb{F} \neq 2$, there is at least one multiple of w which we yields an invertible $a + swc$. Note that we have only assumed that our matrix lies in $\widehat{GSp}_A\left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$, so that the last assertion of Lemma 9.2 implies that $\widehat{GSp}_A\left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right) = GSp_A\left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$ whenever A is not split.

Consider now the case $B = M_2(\mathbb{F})$ and $A = M_4(\mathbb{F})$. By an argument similar to the one given above, we may restrict attention to the case where a is $\begin{pmatrix} I_k & 0 \\ 0 & 0 \end{pmatrix}$ for some $1 \leq k \leq 3$. If $k = 2$ then $c = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$ once again (with entries from $M_2(\mathbb{F})$), the GSp condition imply $\delta = 0$, $\gamma \in \mathbb{F}I$, and $\beta \in GL_2(\mathbb{F})$, and the argument from the case of B division works equally well. In particular, all these elements (as well as the elements containing invertible entries) lie in $GSp_A\left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$. We now claim that the cases $k = 3$ and $k = 1$ can only occur for elements of $\widehat{GSp}_A\left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$ which are not in $GSp_A\left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$. First we demonstrate the existence of elements of $\widehat{GSp}_A\left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$ which are not in $GSp_A\left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$: One example is the matrix $\begin{pmatrix} e & f \\ g & h \end{pmatrix}$ in which $e = \begin{pmatrix} I_3 & 0 \\ 0 & 0 \end{pmatrix}$, $h = \begin{pmatrix} 0 & 0 \\ 0 & I_3 \end{pmatrix}$, f has only lower left entry 1 and other 15 entries vanish, and g has upper right entry 1 (and the rest 0), with multiplier 1 and determinant -1 . Consider now the case $k = 3$, and write $\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$ with the entries from $M_2(\mathbb{F})$. The condition that $\bar{a}c = \begin{pmatrix} 0 & 0 \\ 0 & I_3 \end{pmatrix} \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$ lies in A^- implies, in particular, that in the rightmost column of c the only entry which may be non-zero is the upper right one, which we denote by t . t may not vanish, for otherwise the matrix $\bar{d}a + \bar{b}c = mI$ would have to be singular, a contradiction. But now left multiplication from our representative of $\widehat{GSp}_A\left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)/GSp_A\left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$ would yield a matrix in $M_2(A)$ whose upper left entry is $\begin{pmatrix} I_3 & 0 \\ * & t \end{pmatrix}$ (where $(* \ t)$ is the upper row of c). As this matrix is invertible, the resulting element lies in $GSp_A\left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$, hence the original one $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ does not. In the case $k = 1$, if none of the entries are invertible then c must have rank 3 (again, for $\bar{d}a + \bar{b}c = mI$ to be non-singular), and again we are in the case $k = 3$. This completes the proof of the lemma. \square

Corollary 9.4. *Any element of $GSp_A\left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$, with multiplier m , can be written as $\begin{pmatrix} 1 & v \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ \beta & 1 \end{pmatrix} \begin{pmatrix} a & 0 \\ 0 & m\bar{a}^{-1} \end{pmatrix} \begin{pmatrix} 1 & \alpha \\ 0 & 1 \end{pmatrix}$ for some $a \in A^\times$, and u , α , and β from A^- .*

Proof. This follows directly from Lemmas 9.2 and 9.3. \square

Many arguments below shall make use of the following

Lemma 9.5. *For η and ω in A^- , define $D(\eta, \omega) = 1 + 2\langle \eta, \tilde{\omega} \rangle + |\omega|^2|\eta|^2$. Then the element $1 + \eta\omega$ of A^- has reduced norm $D(\eta, \omega)^2$, and its product with $1 + \tilde{\omega}\tilde{\eta}$ (from either side) yields the scalar $D(\eta, \omega)$.*

Proof. The fact that the products $(1 + \eta\omega)(1 + \tilde{\omega}\tilde{\eta})$ and $(1 + \tilde{\omega}\tilde{\eta})(1 + \eta\omega)$ both yield $D(\eta, \omega)$ easily follow from Lemma 5.2. For the reduced norm, multiply our element by $\tilde{\omega}$ from the right. The result is $\tilde{\omega} + |\omega|^2\eta$ (Lemma 5.2 again) and lies in A^- , so that its reduced norm is $|\tilde{\omega} + |\omega|^2\eta|^2$ by Corollary 5.4. Moreover, Lemma

2.1 evaluates this vector norm as $|\omega|^2 D(\eta, \omega)$, so that this proves the assertion for anisotropic ω . Assume now $|\omega|^2 = 0$. Take some $\xi \in A^-$ with $|\xi|^2 \neq 0$, and consider the two expressions $N_{\mathbb{F}}^A(1 + \eta(\omega + s\xi))$ and $D(\eta, (\omega + s\xi))^2$ for $s \in \mathbb{F}$. Both are polynomials of degree 4 in s , which were seen to coincide wherever $|(\omega + s\xi)|^2 \neq 0$. The latter assumption occurs for any s other than $s = 0$ or $s = -\frac{2\langle\omega, \xi\rangle}{|\xi|^2}$ (Lemma 2.1 again and the isotropy of ω), so that we omit at most two values. By extending scalars if necessary, we may assume that \mathbb{F} has more than 6 elements. Then we have two polynomials of degree 4 which coincide on more than 4 elements of \mathbb{F} , hence they must be the same polynomial. Substituting $s = 0$ in the two equals polynomials verify the assertion for isotropic ω as well. This completes the proof of the lemma. \square

The freedom of choice we have in Lemma 9.3 shows that there are many different choices of parameters to get the same element of $GS p_A \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ in the form of Corollary 9.4. Hence some compatibility assertions will be required wherever we use the form from that Corollary. these will be based on the following

Lemma 9.6. *Assume that the expression using a, v, α and β in Corollary 9.4 yields the same element (of multiplier m) as the one arising from c, w, γ , and δ respectively. Then we have the equalities (i) $c = (1 - (w - v)\beta)a$, (ii) $\delta = \frac{\beta - |\beta|^2(\bar{w} - \bar{v})}{D(\beta, w - v)}$, and (iii) $\gamma = \alpha - ma^{-1} \frac{w - v - |w - v|^2 \bar{\beta}}{D(\beta, w - v)} \bar{a}^{-1}$.*

Proof. The product from Corollary 9.4 equals $\begin{pmatrix} (1+v\beta)a & (1+v\beta)a\alpha + mv\bar{a}^{-1} \\ \beta a & \beta a\alpha + m\bar{a}^{-1} \end{pmatrix}$ (matrix multiplication). When we compare this matrix with the one arising from c, w, γ , and δ , we first find that $\beta a = \delta c$ and $(1 + v\beta)a = (1 + w\delta)c$. Multiplying the first equality by w from the left and subtracting the result from the second equality establishes (i). We write $\delta = \beta a c^{-1}$, substitute c from part (i), use Lemma 9.5 for the inverse of $1 - (w - v)\beta$, and apply Lemma 5.2, which proves part (ii). We can now write $a\alpha$ as the upper right entry of our common matrix minus v times the lower right entry and the same for $c\gamma$ (but with w). Comparing yields $c\gamma = a\alpha - (w - v)(\beta a\alpha + m\bar{a}^{-1})$, which equals $c\alpha - m(w - v)\bar{a}^{-1}$. Multiplying by c^{-1} from the left and using part (i) and Lemmas 9.5 and 5.2 again yields part (iii). This proves the lemma. \square

In addition, the description of the product in the parameters from Corollary 9.4 is given in the following

Lemma 9.7. *Given two elements g and h of $GS p_A \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, with multipliers m and n respectively, assume that $v \in A^-$ is such that left multiplication of both g and gh by $\begin{pmatrix} 1 & -v \\ 0 & 1 \end{pmatrix}$ yields matrices with invertible upper left entry. Then if a, v, α and β are the parameters thus obtained for g as in Corollary 9.4 and e, z, κ , and ν are parameters for h , then parameters for gh may be taken as x, v, ξ , and ζ with $x = a(1 + (\alpha + z)\nu)e$, $\xi = \kappa + ne^{-1} \frac{\alpha + z + |\alpha + z|^2 \bar{\nu}}{D(\alpha + z, \nu)} \bar{e}^{-1}$, and $\zeta = \beta + m\bar{a}^{-1} \frac{\nu + |\nu|^2(\bar{\alpha} + \bar{z})}{D(\alpha + z, \nu)} a^{-1}$.*

Proof. Comparing the expressions for the product shows that the expression $\begin{pmatrix} a & 0 \\ 0 & m\bar{a}^{-1} \end{pmatrix} \begin{pmatrix} 1 & \alpha+z \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ \nu & 1 \end{pmatrix} \begin{pmatrix} e & 0 \\ 0 & n\bar{e}^{-1} \end{pmatrix}$ equals $\begin{pmatrix} 1 & 0 \\ \zeta-\beta & 1 \end{pmatrix} \begin{pmatrix} x & 0 \\ 0 & mn\bar{x}^{-1} \end{pmatrix} \begin{pmatrix} 1 & \xi-\kappa \\ 0 & 1 \end{pmatrix}$. When we consider the upper left entries of both sides we obtain the asserted value for x . The values for ξ (resp. ζ) is obtained by comparing the upper right (resp. lower left) entries, using the value of x , Lemma 9.5, and Lemma 5.2. This proves the lemma. \square

Now, the group which will end up being the Gspin group is, in general, not the full group $GSp_A \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, but a double cover of a certain subgroup. This subgroup is defined using the following

Proposition 9.8. *The map φ which takes an element of $GSp_A \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, decomposes it as in Corollary 9.4, and sends it to the image of $N_{\mathbb{F}}^A(a)$ in $\mathbb{F}^\times / (\mathbb{F}^\times)^2$ is a well-defined group homomorphism $\varphi : GSp_A \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \rightarrow \mathbb{F}^\times / (\mathbb{F}^\times)^2$.*

Proof. Given two decompositions of the same element of $GSp_A \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, part (i) of Lemma 9.6 shows that the reduced norms used to define φ in these decompositions differ by the reduced norm of an element of the form considered in Lemma 9.5. Hence, by that lemma, the result in $\mathbb{F}^\times / (\mathbb{F}^\times)^2$ is the same for both decompositions. Hence φ is well-defined. Now, given two elements of $GSp_A \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, the level of freedom in Lemma 9.3 allows us to find $v \in A^-$ for which left multiplication by $\begin{pmatrix} 1 & -v \\ 0 & 1 \end{pmatrix}$ renders the upper left entries of both g and gh invertible. We then invoke Lemma 9.7, and using Lemma 9.5 for the reduced norm of the multiplier between a and e in x there shows that φ is also multiplicative. This proves the lemma. \square

Note that the choice of the upper left entry in Proposition 9.8 is arbitrary, but does not affect the value of φ in the sense that a similar definition in terms of another entry would yield the same result. Indeed, φ is a group homomorphism (Proposition 9.8) which attains 1 on the unipotent matrices and $GSp_A \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ contains the element $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, which has multiplier 1. As the latter element may be obtained from the parameters $v = -u$, $a = u$, and $\alpha = \beta = \frac{\bar{u}}{|u|^2}$ in Corollary 9.4 for some anisotropic u (by Lemma 5.2), Corollary 5.4 shows that $\varphi \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ is also trivial. It thus suffices to compare the reduced norms of the entries b , c , and d in the matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ when given in the form of Lemma 9.2, and see that if one of them is invertible then it has the same reduced norm as a up to $(\mathbb{F}^\times)^2$. For b and c the assertion follows directly from Corollary 5.4, while $\frac{\bar{a}d}{m}$ (or $\frac{d\bar{a}}{m}$) is an element of the form given in Lemma 9.5. Hence φ is more intrinsic than it seems at first sight. In particular, in case an element of $GSp_A^{\mathbb{F}^2} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ has any invertible entry, we may use the reduced norm of that entry in order to evaluate the φ -image of that element.

Denote the kernel of the map φ from Proposition 9.8 by $GSp_A^{\mathbb{F}^2} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$. We now construct a certain group automorphism of a double cover of the subgroup $GSp_A^{\mathbb{F}^2} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, which is again based on some choice of square root. For the definition we shall use

Lemma 9.9. *Let $a, v, \alpha, \beta, c, w, \gamma$, and δ as in Lemma 9.6, and assume that the (common) element lies in $GSp_A^{\mathbb{F}^2} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$. Let $t \in \mathbb{F}^\times$ be such that $N_{\mathbb{F}}^A(a) = t^2$, and denote $tD(\beta, w - v)$ by s . The following expressions remain invariant by replacing a by c , v by w , α by γ , β by δ , and t by s : (i) $t\tilde{\beta}\bar{a}^{-1}$. (ii) $t(1 + \tilde{v}\beta)\bar{a}^{-1}$. (iii) $t\tilde{\beta}\bar{a}^{-1}\tilde{\alpha} + \frac{m}{t}a$. (iv) $t(1 + \tilde{v}\beta)\bar{a}^{-1}\tilde{\alpha} + \frac{m}{t}\tilde{v}a$.*

Proof. Lemma 9.5 and part (i) of Lemma 9.6 yield $\bar{c}^{-1} = \frac{1 - (\tilde{w} - \tilde{v})\tilde{\beta}}{D(\beta, w - v)}\bar{a}^{-1}$. Part (i) then follows from the definition of s , Lemma 9.5, and part (ii) of Lemma 9.6. Part (ii) is now obtained from part (i), the latter equation, the definition of s , and simple algebra. Now, part (iii) of Lemma 9.6, Lemma 5.6, and the assumption on t imply $s\bar{c}^{-1}\tilde{\gamma} = t(1 - (\tilde{w} - \tilde{v})\tilde{\beta})^{-1}\tilde{\alpha} - \frac{m}{t}(\tilde{w} - \tilde{v})a$. Part (iii) is established using part (ii) of Lemma 9.6, Lemma 9.5, and the definition of s . Part (iv) now follows from the latter equality and part (iii). This completes the proof of the lemma. \square

By definition, an element of $GSp_A \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ belongs to $GSp_A^{\mathbb{F}^2} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ if and only if when decomposed as in Corollary 9.4, the diagonal matrix has entries from $A^{(\mathbb{F}^\times)^2}$. Using the double cover $\tilde{A}^{(\mathbb{F}^\times)^2}$ from Lemma 5.7 we now have

Theorem 9.10. *The group $GSp_A^{\mathbb{F}^2} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ admits a well-defined double cover $\widetilde{GSp}_A^{\mathbb{F}^2} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, in which the parameter $a \in A^{(\mathbb{F}^\times)^2}$ from Corollary 9.4 is replaced by an element $(a, t) \in \tilde{A}^{(\mathbb{F}^\times)^2}$ lying over it. Define a map ψ from $\widetilde{GSp}_A^{\mathbb{F}^2} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ to itself by replacing the parameter (a, t) by $(t\bar{a}^{-1}, t)$ and sending the other parameters from Corollary 9.4 to their θ -images. Then ψ is a well-defined group automorphism of order 2 of $\widetilde{GSp}_A^{\mathbb{F}^2} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, which commutes with the multiplier map to \mathbb{F}^\times .*

Proof. Assume, as in Lemma 9.6, that two sets of parameters in Corollary 9.4, say a, v, α and β versus c, w, γ , and δ , describe the same element of $GSp_A^{\mathbb{F}^2} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$. Lemma 9.5 and part (i) of Lemma 9.6 show that the same element of the double cover $\widetilde{GSp}_A^{\mathbb{F}^2} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ is obtained from (a, t) , v , α and β and from $(c, D(v - w, \beta)t)$, w , γ , and δ . Hence this double cover is well-defined. Consider now our element of $GSp_A^{\mathbb{F}^2} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ written in terms of the parameters (a, t) , v , α and β . Its matrix form appears at the beginning of the proof of Lemma 9.6, and one sees that its ψ -image (using these parameters) has precisely the four entries which appear in the various parts of Lemma 9.9. But this lemma precisely shows that taking the set of parameters $(c, D(v - w, \beta)t)$, w , γ , and δ instead yields the same entries.

This shows that ψ is well-defined, and its image is clearly in $\widetilde{GSp}_A^{\mathbb{F}^2} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ again and has the same multiplier. It is a map of order 2 since so are θ and the map $(a, t) \mapsto (t\bar{a}^{-1}, t)$ on $\tilde{A}^{(\mathbb{F}^\times)^2}$. Finally, let (e, r) , z , κ , and ν be parameters of another element of $\widetilde{GSp}_A^{\mathbb{F}^2} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ and assume that v represents a parameter also for the product of these two elements. Then Lemma 9.7 provides expressions for the parameters x, v, ξ , and ζ of the product in $GSp_A^{\mathbb{F}^2} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, and Lemma

9.5 shows that we may replace x by $(x, tD(a+z, \nu)r)$ for the parameter of the product in the double cover $\widetilde{GSp}_A^{\mathbb{F}^2} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$. Showing that ψ is multiplicative amounts to verifying that sending (a, t) to $(t\bar{a}^{-1}, t)$, (e, r) to $(r\bar{e}^{-1}, r)$, and the A^- -parameters $v, \alpha, \beta, z, \kappa$, and ν to their θ -images results in the same effect on $(x, tD(a+z, \nu)r)$ and on ξ and ζ (the parameter v of the product already appears). Now, for x this follows directly from Lemma 9.5 and the multiplicativity of $y \mapsto \bar{y}^{-1}$ on A^\times , and for ξ and ζ this follows from Lemma 5.6, the assumptions on r and t , the preservation of multipliers, and the fact that θ preserves the bilinear form on A^- (this is relevant also for the action on the denominators $D(\eta, \omega)$). This completes the proof of the theorem. \square

As with φ , the automorphism ψ from Theorem 9.10 may be defined using the other entries, hence is a more intrinsic automorphism of $\widetilde{GSp}_A^{\mathbb{F}^2} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ that one might think at first. To see this, observe that the element $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, of multiplier 1, equals its ψ -image with the appropriate choice of square root: Indeed, it may be obtained from a set of parameters arising from an anisotropic vector u , and the result is independent of u . By taking the square root $-|u|^2$ for the reduced norm of $a = u$ (Corollary 5.4 again), we find that ψ just replaces every instance of u by \tilde{u} , and the resulting matrix must therefore be $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ again. Hence we may use the same argument as for φ in order to obtain that ψ may be evaluated, for example, by applying $(g, t) \mapsto (t\bar{g}^{-1}, t)$ to any invertible entry of the matrix in question. However, we shall stick to our form of Corollary 9.4 in what follows.

The relation between the group $GSp_A^{\mathbb{F}^2} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ and the space $A^- \oplus H$ from Lemma 9.1 (embedded in $M_2(A)$ as described above) begins to reveal itself in the following

Lemma 9.11. *Any anisotropic vector $U \in A^- \oplus H$ lies also in $GSp_A^{\mathbb{F}^2} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, with multiplier $|U|^2$. The involution $\hat{\psi} = Id_H \oplus \theta$ of $A^- \oplus H$ coincides, on anisotropic elements, with the map ψ from Theorem 9.10, after an appropriate lift of these elements into the double cover $\widetilde{GSp}_A^{\mathbb{F}^2} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$. The products $U\hat{\psi}(U)$ and $\hat{\psi}(U)U$ both equal the scalar $|U|^2$, and for pairing of two vectors U and V in $A^- \oplus H$ we have $2\langle U, V \rangle = U\hat{\psi}(V) + V\hat{\psi}(U) = \hat{\psi}(U)V + \hat{\psi}(V)U$.*

Proof. For the first assertion we evaluate $\begin{pmatrix} u & -p \\ q & -\tilde{u} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} \tilde{u} & p \\ -q & -u \end{pmatrix}$ (recall that u and \tilde{u} are in A^-), and the result is indeed $|U|^2 \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ for $U = \begin{pmatrix} u & -p \\ q & -\tilde{u} \end{pmatrix}$. In case $p \neq 0$, we multiply U by $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ (the simplest element with multiplier -1 , clearly equals its ψ -image) and by $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ from the right. The resulting element has multiplier $pq - |u|^2$, and it may be obtained by taking the parameters $a = p$ (with the square root p^2 of $N_{\mathbb{F}}^A(p) = p^4$), $v = 0$, $\alpha = \frac{u}{p}$, and $\beta = \frac{\tilde{u}}{p}$. As $p^2\bar{p}^{-1} = p$ once again, ψ just applies θ to the coordinates u and \tilde{u} , and multiplying by $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ (which was seen to equal its ψ -image) from the right again proves the assertion for the case $p \neq 0$. Otherwise $|u|^2 \neq 0$, and we choose the parameters $v = \alpha = 0$, $a = u$ (for the reduced norm of which Corollary

5.4 allows us to take $-|u|^2$ as a square root), and $\beta = \frac{q\tilde{u}}{|u|^2}$. The resulting ψ -image is once again obtained by just applying θ to u and to \tilde{u} , completing the verification of this assertion. Now, the products $U\hat{\psi}(U) = \begin{pmatrix} u & -p \\ q & -\tilde{u} \end{pmatrix} \begin{pmatrix} \tilde{u} & -p \\ q & -u \end{pmatrix}$ and $\hat{\psi}(U)U = \begin{pmatrix} \tilde{u} & -p \\ q & -u \end{pmatrix} \begin{pmatrix} u & -p \\ q & -\tilde{u} \end{pmatrix}$ are easily evaluated, by Lemma 5.2, as $|U|^2 I$, and the last assertion now follows from Lemma 2.1. This proves the lemma. \square

More importantly, we also have

Proposition 9.12. *If $U \in A^- \oplus H$ and $(g, \psi(g)) \in \widetilde{GSp}_A^{\mathbb{F}^2} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ then the matrix $gU\psi(g)^{-1}$ also lies in $A^- \oplus H$, and it has the same vector norm as U .*

Proof. It suffices to prove the assertion for a generating subset of $\widetilde{GSp}_A^{\mathbb{F}^2} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$. By Corollary 9.4, the set of diagonal and unipotent matrices is a generating set. ψ operates as θ on the A^- -coordinates of the unipotent generators (by choosing 1 to be the square root of $N_{\mathbb{F}}^A(1)$), while on the diagonal ones it operates as $(a, t) \mapsto (t\bar{a}^{-1}, t)$. The composition with inversion as $-\theta$ on the unipotent ones and $(a, t) \mapsto (\frac{\bar{a}}{t}, \frac{1}{t})$ on the diagonal ones. The action of $\begin{pmatrix} 1 & v \\ 0 & 1 \end{pmatrix}$ thus leaves q invariant, takes u to $u + qv$ and \tilde{u} to $\tilde{u} + q\tilde{v}$, and p to $p + u\tilde{v} + v\tilde{u} + qv\tilde{v}$. $\begin{pmatrix} 1 & 0 \\ w & 1 \end{pmatrix}$ sends q to $q + wu + \tilde{u}\tilde{w} + pw\tilde{w}$, u to $u + p\tilde{w}$, \tilde{u} to $\tilde{u} + pw$, and leaves p invariant. Finally, applying $\begin{pmatrix} a & 0 \\ 0 & m\bar{a}-1 \end{pmatrix}$ multiplies q by $\frac{m}{t}$ and p by $\frac{t}{m}$, and maps u to $\frac{au\bar{a}}{t}$, \tilde{u} to $t\bar{a}^{-1}\tilde{u}a^{-1}$. The image of u is the the image of \tilde{u} under θ in all these cases (this is clear in the first two operations and uses Lemma 5.6 and the fact that $t^2 = N_{\mathbb{F}}^A(a)$ for the latter case). In addition, the expressions we add to p in the first case and q in the second case are $2\langle u, v \rangle + q|v|^2$ and $2\langle u, \tilde{w} \rangle + p|w|^2$ by Lemma 5.2 respectively, multiplying which by q (resp. p) yields $|u + qv|^2 - |u|^2$ (resp. $|u + p\tilde{w}|^2 - |u|^2$) by Lemma 2.1. The fact that Lemma 5.7 implies that $|\frac{au\bar{a}}{t}|^2 = |u|^2$ for the latter generators now completes the verification of both assertions for all the necessary cases. This proves the proposition. \square

We shall also make use of the following

Lemma 9.13. *Let $U \in A^- \oplus H$ and $(g, \psi(g)) \in \widetilde{GSp}_A^{\mathbb{F}^2} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ be given. Then the equality $\hat{\psi}(gU\psi(g)^{-1}) = \psi(g)\hat{\psi}(U)g^{-1}$ holds.*

Proof. First, Proposition 9.12 shows that $gU\psi(g)^{-1} \in A^- \oplus H$ and it has vector norm $|U|^2$. In particular, its $\hat{\psi}$ -image is defined. Now, using Lemma 9.11 we write

$$gU\psi(g)^{-1}\hat{\psi}(gU\psi(g)^{-1}) = |gU\psi(g)^{-1}|^2 = |U|^2 = g|U|^2g^{-1}$$

(since $|U|^2$ is a scalar), and applying Lemma 9.11 again the latter term can be presented as $gU\hat{\psi}(U)g^{-1} = gU\psi(g)^{-1} \cdot \psi(g)\hat{\psi}(U)g^{-1}$. If U is anisotropic (hence invertible) then so is $gU\hat{\psi}(U)g^{-1}$, and the assertion follows for such U . For the rest we observe that both sides are linear in U and $A^- \oplus H$ is spanned by anisotropic vectors. This completes the proof of the lemma. \square

We can now give more details to the group action in this case:

Lemma 9.14. *The group $\widetilde{GSp}_A^{\mathbb{F}^2} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ maps to $O(A^- \oplus H)$ with kernel \mathbb{F}^\times , in which the choice of the square root of the reduced norm of the coordinate r of rI is r^2 . Let $\tilde{\psi}$ be an element generating a group of order 2. If $\tilde{\psi}$ operates on $\widetilde{GSp}_A^{\mathbb{F}^2} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ as the automorphism ψ then sending it to the involution $\hat{\psi}$ on $A^- \oplus H$ yields a map from the semi-direct product of $\{1, \tilde{\psi}\}$ with $\widetilde{GSp}_A^{\mathbb{F}^2} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ to $O(A^- \oplus H)$.*

Proof. Proposition 9.12 defines a map $\widetilde{GSp}_A^{\mathbb{F}^2} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \rightarrow O(A^- \oplus H)$. Given $r \in \mathbb{F}^\times$, the scalar matrix rI lies in $\widetilde{GSp}_A^{\mathbb{F}^2} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ (with multiplier r^2), and one easily verifies that it equals its ψ -image if the square root of $N_{\mathbb{F}}^A(r) = r^4$ is taken to be r^2 . This defines an embedding of \mathbb{F}^\times into $\widetilde{GSp}_A^{\mathbb{F}^2} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, with image in the kernel of the action on $A^- \oplus H$ by the centrality of such rI . In order to show that these are the only elements operating trivially, let $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ be an element of $\widetilde{GSp}_A^{\mathbb{F}^2} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, with multiplier m , and let $\begin{pmatrix} e & f \\ g & h \end{pmatrix}$ describe the inverse of its ψ -image (with multiplier $\frac{1}{m}$). The action sends the elements $\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$ and $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ of $A^- \oplus H$ to $\begin{pmatrix} be & bf \\ de & df \end{pmatrix}$ and $\begin{pmatrix} ag & ah \\ cg & ch \end{pmatrix}$ respectively. If the action is trivial, we must have $be = bf = 0$ and $cg = ch = 0$. But then we get, from the GSp relations for $\begin{pmatrix} e & f \\ g & h \end{pmatrix}$, the equalities $b = mb(e\bar{h} + f\bar{g}) = 0$ and $c = mc(h\bar{e} + g\bar{f}) = 0$, so that $a \in A^{(\mathbb{F}^\times)^2}$ with $N_{\mathbb{F}}^A(a) = t^2$, $d = m\bar{a}^{-1}$, $f = g = 0$, $e = \frac{\bar{a}}{t}$, and $h = \frac{ta^{-1}}{m}$. But the action on $A^- \subseteq A^- \oplus H$ was seen in Proposition 9.12 to be via the map from Lemma 5.7, which shows that the only elements $(a, t) \in \tilde{A}^{(\mathbb{F}^\times)^2}$ which act trivially are of the form (r, r^2) with $r \in \mathbb{F}^\times$. In order for the action of this element to be trivial also on $H \subseteq A^- \oplus H$, the formula from Proposition 9.12 implies that m must be r^2 as well, so that our element is indeed rI with ψ -image also rI (note that using the other sign for the ψ -image yields elements operating as $-Id_{A^- \oplus H}$). The fact that $\hat{\psi}$ clearly lies in $O(A^- \oplus H)$ and the scalar $\frac{1}{m(g)}$ operates trivially now implies, together with Lemma 9.13, that the map to $O(A^- \oplus H)$ is well-defined on the semi-direct product in question. This proves the lemma. \square

Once again, we shall need an assertion about reflections:

Lemma 9.15. *An anisotropic vector $g \in A^- \oplus H$ may be lifted to $\widetilde{GSp}_A^{\mathbb{F}^2} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ such that its ψ -image is $-\hat{\psi}(g)$. The map taking $U \in A^- \oplus H$ to the image of $\hat{\psi}(U)$ under the action of this lift of g is the reflection in g .*

Proof. By Lemma 9.11, such g lies in $\widetilde{GSp}_A^{\mathbb{F}^2} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, and $\psi(g)$ can be taken from $\{\pm \hat{\psi}(g)\}$. Hence such a lift to $\widetilde{GSp}_A^{\mathbb{F}^2} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ exists, and operates orthogonally on $A^- \oplus H$ by Proposition 9.12 (or Lemma 9.14). In the evaluation of the result on $U = g$ the two factors involving $\hat{\psi}(g)$ cancel to give just $-g$. On the other hand, if $u \in g^\perp$ then Lemma 9.11 allows us to replace $g\hat{\psi}(U)$ by $-U\hat{\psi}(g)$,

and a similar argument shows that the final result is just U . This proves the lemma. \square

We can finally prove

Theorem 9.16. *The group $\widetilde{GSp}_A^{\mathbb{F}^2} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ is the Gspin group $Gspin(A^- \oplus H)$. It is generated by lifts of anisotropic elements $A^- \oplus H$ whose ψ -images coincide with their $-\hat{\psi}$ -images, so that $GSp_A^{\mathbb{F}^2} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ is generated by $(A^- \oplus H) \cap GL_2(A)$. The spin group $spin(A^- \oplus H)$ is the double cover $\widetilde{Sp}_A^{\mathbb{F}^2} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ of $Sp_A^{\mathbb{F}^2} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ defined by those pairs in $\widetilde{GSp}_A^{\mathbb{F}^2} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ having multiplier 1.*

Proof. Lemma 9.15 and Proposition 2.2 show, as in all the previous cases, that the map from the semi-direct product of Lemma 9.14 to $O(A^- \oplus H)$ is surjective. Moreover, $\hat{\psi}$ has determinant -1 as an element of $O(A^- \oplus H)$ (it inverts a 3-dimensional subspace and leaves the elements of its orthogonal complement invariant), so that $\widetilde{GSp}_A^{\mathbb{F}^2} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ maps to $SO(A^- \oplus H)$ (Lemma 9.15 again), with kernel \mathbb{F}^\times , and the map is again surjective (by index considerations). This shows that $Gspin(A^- \oplus H) = \widetilde{GSp}_A^{\mathbb{F}^2} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, and the structure of the semi-direct product shows (using Lemma 9.15 and Proposition 2.2 again) that this group is generated by those elements of $\widetilde{GSp}_A^{\mathbb{F}^2} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ lying over $(A^- \oplus H) \cap GL_2(A)$ whose images under ψ and $-\hat{\psi}$ coincide. The generation of $GSp_A^{\mathbb{F}^2} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ follows, as the projection from the double cover is surjective. As the element $\hat{\psi}$ of $O(A^- \oplus H)$ inverts a subspace of determinant 1, it has spinor norm 1. Hence Lemma 9.15 implies that this lift of invertible $g \in A^- \oplus H$ has spinor norm $|g|^2$, which coincides with the multiplier of this element. As these were seen to be a generating set for $\widetilde{GSp}_A^{\mathbb{F}^2} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, the spinor norm of any element of the latter group is its multiplier (modulo squares). The fact that this map factors through the projection to $GSp_A^{\mathbb{F}^2} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ is related to the space $A^- \oplus H$ having discriminant 1, so that multiplying by $-Id_{A^- \oplus H}$ does not affect the spinor norms. Therefore $SO^1(A^- \oplus H)$ consists of the images of those elements whose norm is a square, and multiplication by suitable elements from the kernel \mathbb{F}^\times , we may restrict to elements of multiplier 1. These are the elements of the double cover $\widetilde{Sp}_A^{\mathbb{F}^2} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ of the symplectic group $Sp_A^{\mathbb{F}^2} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ which is defined by the multiplier 1 condition. As the only scalars with multiplier 1 are ± 1 , this is the kernel of the (surjective) map from $\widetilde{Sp}_A^{\mathbb{F}^2} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ onto $SO^1(A^- \oplus H)$, whence the former group is indeed $spin(A^- \oplus H)$. This proves the theorem. \square

Our space $A^- \oplus H$ is already assumed to be isotropic. However, we may consider what happens when it splits more than one hyperbolic plane.

Corollary 9.17. *In case $A^- \oplus H$ splits more than one hyperbolic plane, there is a quaternion algebra B over \mathbb{F} such that the Gspin and spin groups are isomor-*

phic to double covers of the subgroups of $GSp_4(B)$ and $Sp_4(B)$ whose presentation as in Corollary 9.4 (but with $M_2(B)^-$ replaced with $M_2^{Her}(B)$ and the lower right entry of the diagonal generators being $m\iota_B(a)^{-t}$ rather than $m\bar{a}^{-1}$) uses parameters from $GL_2^{(\mathbb{F}^\times)^2}(B) \subseteq GL_2(B)$. If it splits more than two hyperbolic planes, then our space is the direct sum of 4 hyperbolic planes, and our description of the spin group presents it as a double cover of the group $SO^1\left(\begin{smallmatrix} 0 & I \\ I & 0 \end{smallmatrix}\right)$ of the direct sum of 4 hyperbolic planes in two ways, which are inequivalent to one another or to the natural presentation as a double cover of such a group. The Gspin group is a double cover of a subgroup of the general special orthogonal group of the direct sum of 4 hyperbolic planes, again in two inequivalent ways.

Proof. Conjugation by an arbitrary element $\begin{pmatrix} e & f \\ g & h \end{pmatrix} \in GL_2(A)$ takes the group $GSp_A\left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$ to the GSp group of the matrix $\begin{pmatrix} e & f \\ g & h \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} \bar{h} & -\bar{f} \\ -\bar{g} & \bar{e} \end{pmatrix}$, with the same multipliers. φ may be defined through the reverse conjugation, and conjugating any ψ -image by the same matrix yields an order 2 automorphism of a double cover of the kernel of this φ (we may also transfer the action on $A^- \oplus H$ by conjugating its image as well). When we do this with $\begin{pmatrix} e & f \\ g & h \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & R \end{pmatrix}$ for $R \in A^- \cap A^\times$ (and multiplying by the global scalar -1) we get $GSp_A\left(\begin{smallmatrix} R & 0 \\ 0 & R \end{smallmatrix}\right)$. In this case Corollary 5.4 shows that the $GSp_A^{\mathbb{F}^2}$ (or $\ker \varphi$) condition keeps its shape, since this conjugation just multiplies the entries by R or R^{-1} . When the A^- part of $A^- \oplus H$ is also isotropic, Corollary 5.10 shows that we can take $A = M_2(B)$, and we choose the element $R = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$. An application of Lemma 1.3 once on matrices in $M_2(A)$ and another time on the entries from $A = M_2(B)$ shows that $GSp_{M_2(B)}\left(\begin{smallmatrix} R & 0 \\ 0 & R \end{smallmatrix}\right)$ is exactly the group of matrices $g \in M_4(B)$ such that $g\begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix}\iota_B(g)^t$ is some multiple of $\begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix}$. The group $GSp_{M_2(B)}\left(\begin{smallmatrix} R & 0 \\ 0 & R \end{smallmatrix}\right)$ is thus $GSp_4(B)$, and the same for the symplectic groups: $Sp_{M_2(B)}\left(\begin{smallmatrix} R & 0 \\ 0 & R \end{smallmatrix}\right) = Sp_4(B)$. Applying this to the generators appearing in Corollary 9.4 indeed yields the generators appearing in the parentheses, and as the upper left entry is not affected, we get the asserted description of the Gspin and spin groups.

We can also conjugate with $\begin{pmatrix} 1 & 0 \\ 0 & S \end{pmatrix}$ with $S \in A^+$, yielding $GSp_A\left(\begin{smallmatrix} S & 0 \\ 0 & -S \end{smallmatrix}\right)$. If A^- splits more than one hyperbolic plane then $A^- \oplus H$ is the sum of 4 hyperbolic planes (see Corollary 5.10 again), $B = M_2(\mathbb{F})$, and $A = M_4(\mathbb{F})$. We choose S to be the tensor product of $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ with itself, i.e., the matrix $\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in M_4(\mathbb{F})$ in which $\alpha = \delta = 0$ and $\gamma = -\beta = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \in M_2(\mathbb{F})$. By the form of the conjugating matrix, the definition of $\ker \varphi$ remains the same also in this case. After applying Lemma 1.3 twice as in the previous case, plus another time on the entries of $B = M_2(\mathbb{F})$, the group $\widehat{GSp}_A\left(\begin{smallmatrix} S & 0 \\ 0 & -S \end{smallmatrix}\right)$ is seen to be the group of matrices $g \in M_8(\mathbb{F})$ such that $g\begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}g^t$ is a multiple of $\begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}$ (i.e., the *general orthogonal group* of that matrix). $GSp_A\left(\begin{smallmatrix} S & 0 \\ 0 & -S \end{smallmatrix}\right)$ is then the *general special orthogonal group*, in which the determinant is the 4th power of the multiplier, and $Sp_A\left(\begin{smallmatrix} S & 0 \\ 0 & -S \end{smallmatrix}\right)$ is just $SO\left(\begin{smallmatrix} 0 & I \\ I & 0 \end{smallmatrix}\right)$.

We claim that the map φ on $Sp_{M_4(\mathbb{F})}\left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$ corresponds to the spinor norm in this presentation as a special orthogonal group. It suffices to verify this again on a set of generators, and we use those from the proof of Proposition

9.12 once more. Recall that we must take the conjugates of our elements by $\begin{pmatrix} 1 & 0 \\ 0 & S \end{pmatrix}$, and that $S \in M_4(\mathbb{F})^+$ and satisfies $S^2 = I$. This conjugation replaces $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ and $\begin{pmatrix} 1 & 0 \\ w & 1 \end{pmatrix}$ by $\begin{pmatrix} 1 & vS \\ 0 & 1 \end{pmatrix}$ and $\begin{pmatrix} 1 & 0 \\ Sw & 1 \end{pmatrix}$, with vS being in $M_4^{as}(\mathbb{F})$ by two applications of Lemma 1.3 (as in Corollary 5.11), and $Sw = Sw\bar{S} \cdot S$ lies there as well. For $\begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix}$ (we restrict to elements of multiplier 1, since we do not consider “spinor norms for general orthogonal groups” here), the choice of S and Lemma 1.3 imply that after conjugation, \bar{a}^{-1} is replaced by a^{-t} . Now, the unipotent generators (which lie in $\ker \varphi$) are squares in $Sp_A \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ (a square root is obtained by dividing the entry vS or Sw from $M_4^{as}(\mathbb{F})$ by 2). Hence they have trivial spinor norms since the range $\mathbb{F}^\times / (\mathbb{F}^\times)^2$ of the spinor norm has exponent 2. For $\begin{pmatrix} a & 0 \\ 0 & a^{-t} \end{pmatrix}$ with $a \in GL_4(\mathbb{F})$, we recall that the latter group is generated by elementary matrices, which operate only on two of these hyperbolic planes. It thus suffices to consider the operation of $\begin{pmatrix} g & 0 \\ 0 & g^{-t} \end{pmatrix}$ with $g \in GL_2(\mathbb{F})$ on the direct sum of two hyperbolic planes. But Corollaries 4.5 and 4.6 show that the latter space is isometric to $M_2(\mathbb{F})$ with the determinant as the vector norm, the Gspin group being the “equal determinant subgroup” of the product of two copies of $GL_2(\mathbb{F})$. By considering the first hyperbolic plane as generated by $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ and $\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$ and the second one by $\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$ and $\begin{pmatrix} 0 & -1 \\ 0 & 0 \end{pmatrix}$, the action of $\begin{pmatrix} g & 0 \\ 0 & g^{-t} \end{pmatrix}$ becomes the action of the pair consisting of g and $\begin{pmatrix} 1 & 0 \\ 0 & \det g \end{pmatrix}$, and the spinor norm is indeed $\det g$. Thus $Sp_{M_4(\mathbb{F})}^{\mathbb{F}^2} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ is isomorphic to $SO^1 \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}$, and our description of the spin group as $\widetilde{Sp}_{M_4(\mathbb{F})}^{\mathbb{F}^2} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ is once again presented as $spin \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}$. We thus have three representations of this group as a spin group of the direct sum of 4 hyperbolic planes: The original one, the projection onto $Sp_{M_4(\mathbb{F})}^{\mathbb{F}^2} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, and the composition of the latter projection with ψ . These representations are not equivalent, since their kernels, all of order 2, are different: The non-trivial element there is $-I$ with ψ -image $-I$ in the original representation, I with ψ -image $-I$ in the projection, and $-I$ with ψ -image I in the composition. On the other hand, $GSp_{M_4(\mathbb{F})}^{\mathbb{F}^2} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ is a subgroup of the general special orthogonal group of $\begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}$ which is defined by some condition which restricts to the triviality of the spinor norm on $SO \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}$, the Gspin group in question is a double cover of this subgroup, and ψ presents it as a double cover of this subgroup in an inequivalent way (again, the projections have different kernels). This completes the proof of the corollary. \square

Note that the groups from Corollary 9.17 are not $\widetilde{GSp}_A^{\mathbb{F}^2} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, but conjugates of the latter group inside $GL_2(A)$, with given conjugators. This yields a definition for ψ on these groups. However, we shall conjugate this ψ by $\begin{pmatrix} R & 0 \\ 0 & R \end{pmatrix}$ or by $\begin{pmatrix} S & 0 \\ 0 & S \end{pmatrix}$. The formula for the resulting map looks just like that of ψ , but in which \bar{a} is replaced by $\iota_B(a)^t$ or just a^t , while $\theta : v \mapsto \tilde{v}$ on A^- becomes $X \mapsto -adj X$ on $M_2^{Her}(B)$ or $T \mapsto \bar{T}$ on $M_4^{as}(\mathbb{F})$ (both having the property that multiplication of the vector and its image under the involution yields the vector norm). Hence when the groups from Corollary 9.17 are considered, this is the choice of ψ with which they come.

The fact that in the hyperbolic case we get 3 inequivalent 8-dimensional representations of the spin group, in all of which the image is the SO^1 group of the direct sum of 4 hyperbolic planes, is an incarnation of *triality* for this case. Triality exists for more general settings, namely some non-isotropic spaces of dimension 8 and discriminant 1 (see Section 35 of [KMRT] for more details), but our methods here restrict to the isotropic case.

We remark that allowing non-trivial spinor norms in the second case in Corollary 9.17 is in some sense dual to allowing multipliers. We have seen that the Gspin group was mapping to the general special orthogonal group in this case. On the other hand, we may allow the map ψ from Theorem 9.10 to have a free choice of a scalar (not necessarily squaring to the reduced norm of an entry), which would extend the definition of ψ to (an \mathbb{F}^\times -cover of) all of $GSp_A\left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$, not only to elements with trivial φ -image. The group constructed in Theorem 9.10 would then be an \mathbb{F}^\times -cover of $GSp_A\left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$, and the action from Proposition 9.12 and Lemma 9.14 may multiply the bilinear form on $A^- \oplus H$ by a scalar. In the split A case this was seen, when we considered the generators $\left(\begin{smallmatrix} a & 0 \\ 0 & \bar{a}-1 \end{smallmatrix}\right)$ with multiplier 1, to produce elements whose image in the projection to $SO(M_4(\mathbb{F})^- \oplus H)$, as well as in the composition of this projection with ψ , may have arbitrary spinor norms (but the spinor norm does have to be the same for these two maps).

In any case, we may have many alternative descriptions of this picture:

Corollary 9.18. *Let Ξ be an element of $\widetilde{GSp}_A^{\mathbb{F}^2}\left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$, such that $\Xi\psi(\Xi)$ is a scalar $r \in \mathbb{F}^\times$. Define the map $\psi_\Xi : \widetilde{GSp}_A^{\mathbb{F}^2}\left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right) \rightarrow \widetilde{GSp}_A^{\mathbb{F}^2}\left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$ by conjugating ψ by Ξ , i.e., $\psi_\Xi(g) = \Xi\psi(g)\Xi^{-1}$. Let $\hat{\psi}_\Xi$ be the composition of $\hat{\psi}$ with the operation of Ξ on $A^- \oplus H$, and we embed the latter space into $M_2(A)$ by multiplying the image from above by Ξ^{-1} from the right. Then all the assertions from Lemma 9.11 to Theorem 9.16 and Corollary 9.17 hold by replacing every U by $U\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$, ψ by ψ_Ξ , and $\hat{\psi}$ by $\hat{\psi}_\Xi$, up to rescaling the bilinear forms by the scalar r .*

Proof. The assumption that $\Xi\psi(\Xi)$ is central implies that ψ_Ξ again has order 2 as an automorphism of $\widetilde{GSp}_A^{\mathbb{F}^2}\left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$. As an element V of the latter space is $U\Xi^{-1}$ with $U \in A^- \oplus H$ as above, its ψ_Ξ -image equals $\Xi\psi(U)\psi(\Xi)^{-1}\Xi^{-1}$. This coincides with our definition of $\hat{\psi}_\Xi$ on U and the modified embedding, so that $\hat{\psi}_\Xi$ preserves this embedding of $A^- \oplus H$ and is the restriction of a branch of ψ_Ξ . In addition, our assumption on Ξ implies that the latter expression is just $\frac{\Xi\hat{\psi}(U)}{r}$. The original Lemma 9.11 now yields its modified version, with the appropriate rescaling. Furthermore, multiplying our V by $\psi_\Xi(g)^{-1} = \Xi\psi(g)^{-1}\Xi^{-1}$ from the right gives $U\psi(g)^{-1}\Xi^{-1}$. After left multiplication by g , the original Proposition 9.12 implies the modified one. All the rest now follows from these assertions in the same way. This proves the corollary. \square

For example, if we take $\Xi = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ (with $r = 1$) in Corollary 9.18 then $A^- \oplus H$ becomes the space of matrices of the form $\begin{pmatrix} p & u \\ \bar{u} & q \end{pmatrix}$, with the usual p, q ,

and u and with minus the “bi-quaternionic Moore determinant” as the vector norm. The map $\hat{\psi}_\Xi$ interchanges p and q with minus one another and leaves A^- pointwise fixed. In general, all the (equivalent) representations we get in Corollary 9.18 are still based on the map ψ , which is more complicated, hence we shall not present any of them explicitly.

10 Dimension 7, Representing the Discriminant

The spaces we consider here are given in

Lemma 10.1. *The orthogonal complement of a vector in $A^- \oplus H$ of some vector norm $-\delta \neq 0$ has discriminant δ and it contains a vector of norm δ . Any vector space of dimension 7 containing a vector whose vector norm equals the discriminant of the space can be obtained in this manner, up to rescaling .*

We remark that if some vector Q has vector norm which equals the discriminant of the space then it continues to hold after rescalings.

Proof. The discriminant and determinant of $A^- \oplus H$ is 1. As a vector Q of vector norm $-\delta$ spans a space of determinant $-\delta$, the complement in $A^- \oplus H$ has the same determinant $-\delta$, and its discriminant is δ since $(-1)^{7(7-1)/2} = -1$. As a hyperbolic plane contains vectors of any given vector norm, the Witt Cancellation Theorem allows us to find some element of $O(A^- \oplus H)$ taking Q to some element of $H \subseteq A^- \oplus H$. The orthogonal complement in H is generated by a vector of vector norm δ , and its inverse image under the orthogonal map we applied has the same vector norm. Conversely, adding some vector Q which is perpendicular to the total space and such that $|Q|^2$ is the determinant of the space yields a space of discriminant 1. The sum of Q with a vector whose vector norm is the discriminant of the space is then isotropic. Lemma 9.1 now completes the proof of the lemma. \square

In view of Lemma 10.1, we write our space as $A^- \oplus \langle \delta \rangle$, using a generator of the orthogonal complement in H whose norm is δ . Moreover, we embed the space $A^- \oplus H$ into $M_2(A)$ as seen after Lemma 9.1, and we choose Q to be the matrix $\begin{pmatrix} 0 & -\delta \\ 1 & 0 \end{pmatrix}$ (of vector norm $-\delta$). We now have

Lemma 10.2. *Given such A with $\iota_B \otimes \iota_C$ and δ , the groups $Gspin(A^- \oplus \langle \delta \rangle)$ and $spin(A^- \oplus \langle \delta \rangle)$ are, up to isomorphism, the stabilizers of the matrix $\begin{pmatrix} 0 & \delta \\ 1 & 0 \end{pmatrix}$ in the action given in Proposition 9.12 inside the groups $\widetilde{GSp}_A^{\mathbb{F}^2} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ and $\widetilde{Sp}_A^{\mathbb{F}^2} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ respectively. The double cover $\widetilde{GSp}_A^{\mathbb{F}^2} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ splits over $Gspin(A^- \oplus \langle \delta \rangle)$. These groups operate by conjugation on $A^- \oplus \langle \delta \rangle$ if we identify the latter space as the space of matrices of the form $\begin{pmatrix} p & \delta u \\ u & -p \end{pmatrix}$ with $u \in A^-$ and $p \in \mathbb{F}$, with the vector norm being the “bi-quaternionic A^- -Moore determinant” divided by $-\delta$.*

Proof. The first assertion is proved by the same argument used for Lemma 6.2. Now, that the action from Proposition 9.12 is based on the map ψ , hence an element of $GSp_A^{\mathbb{F}^2} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ would stabilize $\begin{pmatrix} 0 & -\delta \\ 1 & 0 \end{pmatrix}$ if and only if the ψ -image of one of its lifts equals its conjugate by $\begin{pmatrix} 0 & -\delta \\ 1 & 0 \end{pmatrix}$. This yields the splitting of the double cover over $Gspin(A^- \oplus \langle \delta \rangle)$, since every element there comes with a natural choice of ψ -image. Take now $\Xi = \begin{pmatrix} 0 & -\delta \\ 1 & 0 \end{pmatrix}$ in Corollary 9.18, which equals its ψ -image and squares to $-\delta$. The space thus obtained is the one written explicitly here, and the remaining assertions follow since $\psi_\Xi(g) = g$ for $g \in Gspin(A^- \oplus \langle \delta \rangle)$ by definition. This proves the lemma. \square

In order to give a more detailed description of the groups from Lemma 10.2, we begin by proving

Lemma 10.3. *If an element of $\widetilde{GSp}_A^{\mathbb{F}^2} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ lying over $\begin{pmatrix} e & f \\ g & h \end{pmatrix} \in GSp_A^{\mathbb{F}^2} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ stabilizes $\begin{pmatrix} 0 & -\delta \\ 1 & 0 \end{pmatrix}$ then either e or g are invertible.*

Proof. First observe that is $(a, t) \in \widetilde{A}^{(\mathbb{F}^\times)^2}$ then the action of the element $\begin{pmatrix} a & 0 \\ 0 & t\bar{a}^{-1} \end{pmatrix}$ of $\widetilde{GSp}_A^{\mathbb{F}^2} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, with ψ -image $\begin{pmatrix} t\bar{a}^{-1} & 0 \\ 0 & ma/t \end{pmatrix}$, stabilizes this matrix. The proof of Lemma 9.3 thus shows that it suffices to consider elements in which $e = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \in A = M_2(B)$ for some quaternion algebra B over \mathbb{F} (for A division the lemma is immediate). We have seen in the proof of Lemma 9.3 that g takes the form $\begin{pmatrix} \lambda & \mu \\ r & 0 \end{pmatrix}$ with λ and μ from B and $r \in \mathbb{F}$, and a similar argument shows that $f = \begin{pmatrix} \sigma & s \\ \tau & 0 \end{pmatrix}$ where σ and τ are in B and $s \in \mathbb{F}$. Moreover, $\mu\bar{\tau}$ equals the multiplier m (hence both μ and τ are invertible), and h has lower right entry $m+rs$. As parameters for $\begin{pmatrix} e & f \\ g & h \end{pmatrix}$ in Corollary 9.4 may be taken to be $v = \begin{pmatrix} 0 & 0 \\ -1 & 0 \end{pmatrix}$, $a = \begin{pmatrix} 1 & 0 \\ \lambda & \mu \end{pmatrix}$, α with upper right entry $s \in \mathbb{F}$, and $\beta = \begin{pmatrix} 0 & 1 \\ r & 0 \end{pmatrix}$. The $GSp_A^{\mathbb{F}^2}$ condition means that $N_{\mathbb{F}}^B(a) = N_{\mathbb{F}}^B(\mu)$ is a square, say t^2 , hence $t\bar{a}^{-1} = \begin{pmatrix} \mu/t & 0 \\ \lambda\mu/t & t \end{pmatrix}$, and the action of θ leaves u , β , and the s -entry of α invariant. The lowest row of $\begin{pmatrix} 1 & v \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ \beta & 1 \end{pmatrix}$ is $(r \ 0 \ 0 \ 1)$, while in $t\bar{a}^{-1} \begin{pmatrix} 1 & \alpha \\ 0 & 1 \end{pmatrix}$ the most upper right and lower right entries are $\frac{s\mu}{t}$ and $\frac{m\mu}{t}$ respectively. Hence the most lower right entry of $\psi \begin{pmatrix} e & f \\ g & h \end{pmatrix}$ is $\frac{(m+rs)\mu}{t}$. As $\psi \begin{pmatrix} e & f \\ g & h \end{pmatrix}$ has multiplier m , its inverse has $\frac{(m+rs)\mu}{mt}$ as its most upper left entry. Now, the second row of $\begin{pmatrix} e & f \\ g & h \end{pmatrix} \begin{pmatrix} 0 & -\delta \\ 1 & 0 \end{pmatrix}$ is $(\tau \ 0 \ 0 \ 0)$, so that the second row of the action of $\begin{pmatrix} e & f \\ g & h \end{pmatrix}$ on $\begin{pmatrix} 0 & -\delta \\ 1 & 0 \end{pmatrix}$ starts with $\frac{(m+rs)\tau\mu}{mt}$. But we have assumed that $\begin{pmatrix} e & f \\ g & h \end{pmatrix}$ preserves $\begin{pmatrix} 0 & -\delta \\ 1 & 0 \end{pmatrix}$, so that the latter expression must vanish. As τ and μ are in B^\times and $m \neq 0$, it follows that $r \neq 0$, whence $g = \begin{pmatrix} \lambda & \mu \\ r & 0 \end{pmatrix}$ is invertible as desired. This completes the proof of the lemma. \square

The determination of the group from Lemma 10.2 may now be carried out using the explicit formulae for ψ . The result is

Proposition 10.4. *Any element of $Gspin(A^- \oplus \langle \delta \rangle)$ may be presented either as $\begin{pmatrix} a & -t\delta\beta\bar{a}^{-1} \\ \beta a & t\bar{a}^{-1} \end{pmatrix}$ with $(a, t) \in \widetilde{A}^{(\mathbb{F}^\times)^2}$ and $\beta \in A^-$, or as $\begin{pmatrix} w c & -s\delta\bar{c}^{-1} \\ c & s\bar{w}\bar{c}^{-1} \end{pmatrix}$ where (c, s) is in $\widetilde{A}^{(\mathbb{F}^\times)^2}$ and w comes from A^- .*

Proof. Lemma 10.3 implies that for every element $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GSp_A^{\mathbb{F}^2} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ which stabilizes $\begin{pmatrix} 0 & -\delta \\ 1 & 0 \end{pmatrix}$, either $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ or $\begin{pmatrix} 0 & -\delta \\ 1 & 0 \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ has an invertible upper right entry. In the first case a lies under some element $(a, t) \in \tilde{A}^{(\mathbb{F}^\times)^2}$, and Lemma 9.2 shows that $b = \beta a$ for some $\beta \in A^-$. Moreover, the formula from Theorem 9.10 shows that $\psi \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ has left entry $\begin{pmatrix} t\bar{a}^{-1} \\ t\beta\bar{a}^{-1} \end{pmatrix}$, and if $\psi \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ coincides with the image of $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ under conjugation by $\begin{pmatrix} 0 & -\delta \\ 1 & 0 \end{pmatrix}$, then $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ must have the asserted right column. If a is not invertible, then we may multiply $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ by $\begin{pmatrix} 0 & -\delta \\ 1 & 0 \end{pmatrix}$ from the left, obtain an element of the form just described, and dividing by $\begin{pmatrix} 0 & -\delta \\ 1 & 0 \end{pmatrix}$ back again shows that our element must be of the second suggested form. This proves the proposition. \square

Note that an element of the second form may be uniquely presented as the product of an anisotropic element of $A^- \oplus \langle \delta \rangle = \begin{pmatrix} 0 & -\delta \\ 1 & 0 \end{pmatrix}^\perp \subseteq A^- \oplus H$ in which the lower left entry is 1 and a diagonal matrix stabilizing $\begin{pmatrix} 0 & \delta \\ 1 & 0 \end{pmatrix}$ (the latter multipliers form, as Proposition 10.4 shows, a group which is isomorphic to $\tilde{A}^{(\mathbb{F}^\times)^2}$): Indeed, such an element is just the product $\begin{pmatrix} w & \delta \\ 1 & -\bar{w} \end{pmatrix} \begin{pmatrix} c & 0 \\ 0 & -r\bar{c}^{-1} \end{pmatrix}$.

In total, we have

Theorem 10.5. *The Gspin group of $A^- \oplus \langle \delta \rangle$ consists of elements $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ of $GSp_A^{\mathbb{F}^2} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ in which $a\bar{d}$ and $b\bar{c}$ are scalars from \mathbb{F} , which square to $N_{\mathbb{F}}^A(a)$ (or equivalently $N_{\mathbb{F}}^A(d)$) and $\delta^2 N_{\mathbb{F}}^A(c)$ (which equals also $\frac{N_{\mathbb{F}}^A(b)}{\delta^2}$) respectively. It is characterized by either the two elements bd^{-1} and $-\delta ca^{-1}$ or the two elements ac^{-1} and $-\delta db^{-1}$ being well-defined elements of A^- which are θ -images of one another. It is generated by anisotropic vectors of the form $\begin{pmatrix} v & \delta \\ 1 & -\bar{v} \end{pmatrix}$ or $\begin{pmatrix} v & 0 \\ 0 & -\bar{v} \end{pmatrix}$. The spin group consists of those elements in which the two scalars $a\bar{d}$ and $b\bar{c}$ sum to 1.*

Proof. Any element may be presented in one of the two forms given in Proposition 10.4. In the first case we have $a\bar{d} = t$ and $b\bar{c} = t\delta|\beta|^2$, while in the second one these numbers are $-s|w|^2$ and $-\delta s$ respectively. The relations with the reduced norms of a, b, c , and d are easily verified using Corollary 5.4 and the definition of $\tilde{A}^{(\mathbb{F}^\times)^2}$, and the relation between bd^{-1} and $-\delta ca^{-1}$ in the first case and ac^{-1} and $-\delta db^{-1}$ in the third case are also immediate. Conversely, if $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is a matrix in which $a\bar{d}$ and $b\bar{c}$ are scalars, not both zero, then either $d = t\bar{a}^{-1}$ or $b = -s\delta\bar{c}^{-1}$ for some scalars t and s . If $t^2 = N_{\mathbb{F}}^A(a)$ then $N_{\mathbb{F}}^A(d)$ takes the same value, while if $s^2 = N_{\mathbb{F}}^A(c)$ then $N_{\mathbb{F}}^A(b)$ is obtained by multiplication by δ^4 . If we write $c = \beta a$ in the first case and $a = wc$ in the second case, then the respective values $b = -t\delta\bar{\beta}\bar{a}^{-1}$ and $d = s\bar{w}\bar{c}^{-1}$ immediately follow. Now, elements with invertible lower left entry were seen to take the form $\begin{pmatrix} w & \delta \\ 1 & -\bar{w} \end{pmatrix} \begin{pmatrix} c & 0 \\ 0 & -s\bar{c}^{-1} \end{pmatrix}$. The generation of these elements by the asserted set now follows from Theorem 5.9, as the map $(a, t) \mapsto \begin{pmatrix} a & 0 \\ 0 & t\bar{a}^{-1} \end{pmatrix}$ is a group injection which sends the generators $(v, |v|^2)$ for $v \in A^- \cap A^\times$ to $\begin{pmatrix} v & 0 \\ 0 & -\bar{v} \end{pmatrix}$. The other elements are obtained by multiplying the appropriate elements by the generator $\begin{pmatrix} 0 & \delta \\ 1 & 0 \end{pmatrix}$, and the assertion about generation follows. For the spin group, observe that the proof of Lemma 6.2

shows that the spinor norm of an element of $SO(A^- \oplus \langle \delta \rangle)$ is the same when considered there or in $SO(A^- \oplus H)$ (by leaving $\begin{pmatrix} 0 & -\delta \\ 1 & 0 \end{pmatrix}$ invariant), and the proof of Theorem 9.16 implies that in the latter group the spinor norms of (the image of) $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is just the multiplier $a\bar{d} + b\bar{c}$. As by the usual scalar multiplication we may normalize this multiplier to 1 wherever it is a square, the assertion about the spin group is also established. This proves the theorem. \square

After fixing A , we have the following assertion about the dependence of the Gspin and spin groups on δ :

Proposition 10.6. *If $\varepsilon \in \delta(\mathbb{F}^\times)^2 N_{\mathbb{F}}^A(A^\times)$ then the spin and Gspin groups of $A^- \oplus \langle \varepsilon \rangle$ are isomorphic to those of $A^- \oplus \langle \delta \rangle$.*

Proof. Consider first multiplication from $(\mathbb{F}^\times)^2$. Let $r \in \mathbb{F}^\times$, and we examine the result of conjugation by $\begin{pmatrix} 1 & 0 \\ 0 & 1/r \end{pmatrix}$. This operation multiplies the upper right entry by r and divides the lower left entry by r . Hence on elements of $Gspin(A^- \oplus \langle \delta \rangle)$ of the first form of Proposition 10.4 this operation corresponds to leaving a (and t) invariant, dividing β by r , and multiplying δ by r^2 , while for elements of the second form it means dividing c by r (hence dividing s by r^2), multiplying w by r , and again multiplying δ by r^2 . Hence this conjugation takes $Gspin(A^- \oplus \langle \delta \rangle)$ into $Gspin(A^- \oplus \langle r^2 \delta \rangle)$. Conjugation by the inverse element shows that the map between these two groups is bijective. As for multiplication by norms from A^\times , we now consider the conjugation by $\begin{pmatrix} e & 0 \\ 0 & \bar{e}^{-1} \end{pmatrix}$ for some $e \in A^\times$. For elements having the first form in Proposition 10.4, this operation sends a to eae^{-1} (hence t remains invariant) and β to $\bar{e}^{-1}\beta e^{-1}$, and Lemma 5.6 shows that δ must be multiplied by $N_{\mathbb{F}}^A(e)$. As for the other elements, this operation takes c to $\bar{e}^{-1}ce^{-1}$ (and therefore s is divided by $N_{\mathbb{F}}^A(e)$) and w to $ew\bar{e}$, so that again δ has to be multiplied by $N_{\mathbb{F}}^A(e)$ (Lemma 5.6 is just a consistency check in this case). This $Gspin(A^- \oplus \langle \delta \rangle)$ is isomorphic to a subgroup of $Gspin(A^- \oplus \langle N_{\mathbb{F}}^A(e)\delta \rangle)$, and an argument using the inverse conjugation show the bijectivity. As conjugation preserves multipliers and the spin groups are the subgroups of the Gspin groups which are defined by the multiplier 1 condition, this completes the proof of the proposition. \square

By Proposition 10.6, it suffices to take δ from a set of representatives for $\mathbb{F}^\times/(\mathbb{F}^\times)^2 N_{\mathbb{F}}^A(A^\times)$. In addition, Lemma 6.6 once again shows that if $A = M_2(B)$ then this group involves just classes modulo $N_{\mathbb{F}}^B(B^\times)$.

The concept of isotropy in this case is the one considered in

Corollary 10.7. *Assume that $A^- \oplus \langle \delta \rangle$ contains an isotropic vector which is orthogonal to a vector of vector norm which equals the discriminant. Then the Gspin and spin group consist of matrices $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ from $GSp_4(B)$ (or $Sp_4(B)$), for some quaternion algebra B over \mathbb{F} , in which $a_L(d)^t$ and $b_L(c)^t$ lie in \mathbb{F} , and square to $N_{\mathbb{F}}^{M_2(B)}(a) = N_{\mathbb{F}}^{M_2(B)}(d)$ and $\delta^2 N_{\mathbb{F}}^{M_2(B)}(c) = \frac{N_{\mathbb{F}}^{M_2(B)}(b)}{\delta^2}$ respectively. In every such matrix, either bd^{-1} and $-\delta ca^{-1}$ or ac^{-1} and $-\delta db^{-1}$ lie in $M_2^{Her}(B)$ and are minus the adjoints of one another. These groups operate by conjugation on the space of matrices $\begin{pmatrix} pI & -\delta X \\ a_{dI}X & -pI \end{pmatrix} \in M_4(B)$, where*

$X \in M_2^{Her}(B)$ and $p \in \mathbb{F}$, as the $Gspin$ and $spin$ groups of this space with the “bi-quaternionic A^- -Moore determinant” divided by $-\delta$ as the vector norm. The $spin$ group may also be presented as the $spin$ group of the space of matrices of the sort $\begin{pmatrix} \delta X & pI \\ pI & adj X \end{pmatrix}$, with the same “bi-quaternionic Moore determinant” as the vector norm, via $g : N \mapsto gN_{\mathcal{L}B}(g)^t$.

In case $A^- \oplus \langle \delta \rangle$ splits three hyperbolic planes, we get groups of 8×8 matrices $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ which multiply the bilinear form defined by $\begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}$ by a scalar, such that ad^t and bc^t are scalar 4×4 matrices, whose squares are $\det a = \det d$ and $\delta^2 \det c = \frac{\det b}{\delta^2}$ respectively. Moreover, either the pair bd^{-1} and $-\delta ca^{-1}$ or the pair ac^{-1} and $-\delta db^{-1}$ are defined, they lie in $M_4^{as}(\mathbb{F})$, and they are sent to one another by the involution $T \mapsto \hat{T}$ which was described in the paragraph following Corollary 5.11. The space on which these groups operate by conjugation consists of matrices of the form $\begin{pmatrix} pI & -\delta T \\ T & -pI \end{pmatrix}$ with $T \in M_4^{as}(\mathbb{F})$.

Proof. We have seen in the proof of Corollary 9.17 that $GSp_4(B)$ is obtained from $GSp_{M_2(B)}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ through conjugation by $\begin{pmatrix} 1 & 0 \\ 0 & R \end{pmatrix}$ with $R = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$. As this operation takes a matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ to $\begin{pmatrix} a & bR^{-1} \\ Rc & RdR^{-1} \end{pmatrix}$, Lemma 1.3 shows that the relations from Theorem 10.5 become the ones asserted here (the reduced norms are not affected, since $N_{\mathbb{F}}^{M_2(B)}(R) = 1$). The assertions involving bd^{-1} and $-\delta ca^{-1}$ or ac^{-1} and $-\delta db^{-1}$ follow from those appearing in Theorem 10.5 through the fact that $\theta(X) = -adj X$ for $X \in M_2(B)^-$, right multiplication by R sends this space to $M_2^{Her}(B)$, and $adj R = R^{-1}$ for our R . Recall now that the group $Gspin(A^- \oplus \langle \delta \rangle) \subseteq GSp_{M_2(B)}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ operates on the space defined in Lemma 10.2 by conjugation. Conjugating the formula for this action by $\begin{pmatrix} 1 & 0 \\ 0 & R \end{pmatrix}$ yields the action of our subgroup of $GSp_4(B)$ by conjugation on the asserted space with the asserted quadratic form. The Sp_4 condition now shows that multiplying the latter space from the right by $\begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix}$ yields a space on which $spin(M_2^{Her}(B) \oplus \langle \delta \rangle) \subseteq Sp_4(B)$ operates via $g : N \mapsto gN_{\mathcal{L}B}(g)^t$, and this space is easily seen to be the one from the last assertion,

In the case of splitting three hyperbolic planes, we apply the same argument with the matrix S from the proof of Corollary 9.17. Once again Lemma 1.3 yields the desired relations between the squares, and $\det S = 1$. We recall from the proof of Corollary 5.11 that multiplication of $M_4(\mathbb{F})^-$ by S (from either side) yields anti-symmetric matrices, and that the vector norm is taken to minus the pfaffian by this operation. Conjugating the space from Lemma 10.2 by $\begin{pmatrix} 1 & 0 \\ 0 & S \end{pmatrix}$ yields the first space with T and \hat{T} . The fact that the $spin$ group is contained in $O\begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}$ allows us to multiply our representation by $\begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}$ from the right, yielding the second space with the action $g : L \mapsto gLg^t$ of the $spin$ group. This completes the proof of the corollary. \square

Note that the description of the groups from Corollary 10.7 is in correspondence with the choice of ψ on the groups from Corollary 9.17. For the $Gspin$ groups, the representations which extend those defined by $g : N \mapsto gN_{\mathcal{L}B}(g)^t$ and $g : L \mapsto gLg^t$ from Corollary 10.7 and preserve the bilinear form must include division by the multiplier m . In addition, although in both cases we may

obtain natural 8-dimensional representations of these groups by adding $\begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix}$ to the first representation and $\begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}$ to the second one, this is not dual to preserving $\begin{pmatrix} 0 & -\delta \\ 1 & 0 \end{pmatrix}$ since the full GSp group (of $\begin{pmatrix} Q & 0 \\ 0 & Q \end{pmatrix}$ or $\begin{pmatrix} S & 0 \\ 0 & -S \end{pmatrix}$) also preserve this matrix by definition. We also mention the fact that starting with the representation appearing in Corollary 9.18 yields precisely the representations of $Gspin(A^- \oplus \langle \delta \rangle)$ and $spin(A^- \oplus \langle \delta \rangle)$ already given in Lemma 10.2 and Corollary 10.7.

11 Dimension 8, Isotropic, Any Discriminant

Let d be a discriminant, and let $\mathbb{E} = \mathbb{F}(\sqrt{d})$ be the associated quadratic extension of \mathbb{F} , with Galois automorphism ρ . We shall be interested in the spaces given in the following

Lemma 11.1. *Let $A = B \otimes C$ be a bi-quaternion algebra over \mathbb{F} with the involution corresponding to this presentation, and let $Q \in A^-$ be anisotropic. The direct sum $(A_{\mathbb{E}}^-)_{\rho, Q} \oplus H$ of a hyperbolic plane and the space from Lemma 7.1 is 8-dimensional, isotropic, and has discriminant d . Moreover, this yields all the isotropic 8-dimensional quadratic spaces of discriminant d over \mathbb{F} .*

Proof. The space $(A_{\mathbb{E}}^-)_{\rho, Q}$ from Lemma 7.1 has dimension 6 and discriminant d , hence determinant $-d$. Adding the isotropic space H , of determinant -1 yields a space with the desired properties. On the other hand, if an 8-dimensional space is isotropic and has discriminant d , then it splits a hyperbolic plane, and the complement has dimension 6 and discriminant d . The lemma now follows from Lemma 7.1 and the fact that hyperbolic planes are isometric to their rescalings. \square

Extending scalars in the space from Lemma 11.1 to \mathbb{E} , we obtain an isotropic 8-dimensional space of discriminant 1, which equals $A_{\mathbb{E}}^- \oplus H_{\mathbb{E}}$ by Lemma 9.1 and we present it as the subspace of $M_2(A_{\mathbb{E}})$ as before. Therefore our space $(A_{\mathbb{E}}^-)_{\rho, Q}$ is isomorphic to the space of matrices $\begin{pmatrix} u & -p \\ q & -u \end{pmatrix}$ in which $u \in (A_{\mathbb{E}}^-)_{\rho, Q}$ and p and q are in \mathbb{F} , with the restriction of the quadratic form from $A_{\mathbb{E}}^- \oplus H_{\mathbb{E}}$. Proposition 9.12 shows that the group $\widetilde{GSp}_{A_{\mathbb{E}}}^{\mathbb{E}^2} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ acts on $A_{\mathbb{E}}^- \oplus H_{\mathbb{E}}$, and we are interested in the subgroup which preserves the subspace $(A_{\mathbb{E}}^-)_{\rho, Q} \oplus H$. Observe that the H part of $(A_{\mathbb{E}}^-)_{\rho, Q} \oplus H$ is invariant under the automorphism ρ of $M_2(A)$, and the action of ρ on the other part is given in Lemma 7.1. In particular ρ preserves the subspace $(A_{\mathbb{E}}^-)_{\rho, Q} \oplus H$ of $A_{\mathbb{E}}^- \oplus H_{\mathbb{E}}$, and operates as the reflection in Q on this space.

We shall be needing also non-invertible elements of A having the $A_{\mathbb{E}, \rho, \mathbb{F}Q}^{t^2}$ -property, namely those $a \in A$ which satisfy $aQ\bar{a}^{\rho} = 0$ (the reduced norm condition immediately follows, since $a \notin A^{\times}$ hence its reduced norm vanishes). We denote the union of the set of those elements with $A_{\mathbb{E}, \rho, \mathbb{F}Q}^{t^2}$ by $A_{\mathbb{E}, \rho, \mathbb{F}Q}^{t^2, 0}$. It is no longer a group, but it is closed under multiplication, and $t : A_{\mathbb{E}, \rho, \mathbb{F}Q}^{t^2, 0} \rightarrow \mathbb{F}$

(including 0) is multiplicative. Moreover, apart from $(A_{\mathbb{E}}^-)_{\rho,Q}$, we shall also be needing the space $(A_{\mathbb{E}}^-)_{\rho,\tilde{Q}}$. We shall need a few simple relations between these sets.

Lemma 11.2. *Let $v \in (A_{\mathbb{E}}^-)_{\rho,Q}$ and $w \in (A_{\mathbb{E}}^-)_{\rho,\tilde{Q}}$ be given. Then we have (i) $\tilde{v} \in (A_{\mathbb{E}}^-)_{\rho,\tilde{Q}}$. (ii) $Q^{-1}vQ^{-1}$ is also in $(A_{\mathbb{E}}^-)_{\rho,\tilde{Q}}$. (iii) $vQ^{-1} \in A_{\mathbb{E},\rho,\mathbb{F}Q}^{t^2,0}$, with $t(vQ^{-1}) = -\frac{|v|^2}{|Q|^2}$. (iv) $\tilde{w} \in (A_{\mathbb{E}}^-)_{\rho,Q}$. (v) QwQ also lies in $(A_{\mathbb{E}}^-)_{\rho,Q}$. (vi) Qw is an element of $A_{\mathbb{E},\rho,\mathbb{F}Q}^{t^2,0}$, whose multiplier $t(Qw)$ is $-|Q|^2|w|^2$.*

Proof. Lemma 7.1 implies $v^\rho = -\frac{Q\tilde{v}Q}{|Q|^2}$, hence $\tilde{v} = -|Q|^2Q^{-1}v^\rho Q^{-1} = -\frac{\tilde{Q}v^\rho\tilde{Q}}{|\tilde{Q}|^2}$ since $Q^{-1} = \frac{\tilde{Q}}{|\tilde{Q}|^2}$ and $|\tilde{Q}|^2 = |Q|^2$. Applying ρ to the latter equation and using the fact that $\theta^2 = Id_{A_{\mathbb{E}}^-}$ and that Q , hence also \tilde{Q} , are ρ -invariant, yields the \tilde{Q} -based condition from Lemma 7.1 for \tilde{v} . This establishes part (i). For part (ii), recall first that ρ preserves the space $(A_{\mathbb{E}}^-)_{\rho,Q}$. It thus preserves also $(A_{\mathbb{E}}^-)_{\rho,\tilde{Q}}$. Write v as $(v^\rho)^\rho$, which equals $-\frac{Q\tilde{v}^\rho Q}{|Q|^2}$ by Lemma 7.1. Hence $Q^{-1}vQ^{-1} = -\frac{\tilde{v}^\rho}{|Q|^2}$, which implies part (ii) since the latter element lies in $(A_{\mathbb{E}}^-)_{\rho,\tilde{Q}}$ by part (i). For part (iii) observe that as Q^{-1} and v lie in $A_{\mathbb{E}}^-$ and $Q^{-\rho} = Q^{-1}$, the expression $vQ^{-1} \cdot Q \cdot \overline{vQ^{-1}^\rho}$ equals $vQ^{-1}v^\rho$. Substituting the expression for v^ρ from Lemma 7.1 again, the latter expression becomes $-\frac{|v|^2}{Q^2}Q$, and part (iii) follows since the reduced norm condition is a consequence of Corollary 5.4. Parts (iv) and (v) are proved either by applying the necessary changes in the proofs of parts (i) and (ii) respectively, or since the maps given in parts (i) and (ii) are injective maps between 6-dimensional vector space over \mathbb{F} and the maps from parts (iv) and (v) are their inverses. For part (vi) we write w as $Q^{-1}uQ^{-1}$ for some $u \in (A_{\mathbb{E}}^-)_{\rho,Q}$ using parts (ii) and (v), and then the assertion for $Qw = uQ^{-1}$ follows from part (iii) since $u = QwQ = -Qw\tilde{Q}$ has vector norm $|Q|^4|w|^2$ by Proposition 5.5 and Corollary 5.4. This completes the proof of the lemma. \square

We have seen that parts (i) and (iv) in Lemma 11.2, as well as parts (ii) and (v) there, are inverses. Moreover, a claim similar to part (iii) (but without the multiplier) appears in the first assertion of Lemma 7.5. We remark that the proof of parts (iii) and (iv) in that lemma shows that if the vector v or w is anisotropic then the converse implication also holds (cancel vQ from the left in the proof of part (iii), and applying the same argument to extend it to part (vi)). On the other hand, if v (or w) are not isotropic then the converse implications in parts (iii) and (iv) of Lemma 11.2 may not hold. Indeed, by taking a non-zero isotropic vector v in $(A_{\mathbb{E}}^-)_{\rho,Q}$ and $z \in \mathbb{E} \setminus \mathbb{F}$, then vQ^{-1} as well as zvQ^{-1} lie in $A_{\mathbb{E},\rho,\mathbb{F}Q}^{t^2,0}$, while zv no longer lies in $(A_{\mathbb{E}}^-)_{\rho,Q}$ (and the same for $w \in (A_{\mathbb{E}}^-)_{\rho,\tilde{Q}}$). Note that this argument does not affect our assertions in the anisotropic case, since in this case zv would have vector norm $z^2|v|^2$ and the multiplier of zvQ^{-1} (which does belong to $A_{\mathbb{E},\rho,\mathbb{F}Q}^\times$) is $-N_{\mathbb{F}}^{\mathbb{E}}(z)\frac{|v|^2}{|Q|^2}$, which do not coincide if $|v|^2 \neq 0$.

We shall also need the following complement of Lemma 9.5 here:

Lemma 11.3. *For $\eta \in (A_{\mathbb{E}}^-)_{\rho,Q}$ and $\omega \in (A_{\mathbb{E}}^-)_{\rho,\tilde{Q}}$, the element $1 + \eta\omega$ of A lies in $A_{\mathbb{E},\rho,\mathbb{F}Q}^{t^2,0}$, with multiplier $D(\eta,\omega)$.*

Proof. First note that the expression $D(\eta,\omega)$ from Lemma 9.5 lies in \mathbb{F} for such η and ω , since $(A_{\mathbb{E}}^-)_{\rho,Q}$ and $(A_{\mathbb{E}}^-)_{\rho,\tilde{Q}}$ are quadratic spaces over \mathbb{F} . Now, as in the proof of Lemma 9.5, we begin by assuming that ω is anisotropic, and write $1 + \eta\omega$ as $(\tilde{\omega} + |\omega|^2\eta) \frac{\omega}{|\omega|^2}$. Then $\frac{\omega}{|\omega|^2} \in (A_{\mathbb{E}}^-)_{\rho,Q}$, and $\tilde{\omega} + |\omega|^2\eta \in (A_{\mathbb{E}}^-)_{\rho,\tilde{Q}}$ by part (iv) of Lemma 11.2. But now parts (iii) and (iv) of that lemma show that $(\tilde{\omega} + |\omega|^2\eta)Q^{-1}$ and $Q \frac{\omega}{|\omega|^2}$ are both in $A_{\mathbb{E},\rho,\mathbb{F}Q}^{t^2,0}$, with multipliers $-\frac{|\tilde{\omega} + |\omega|^2\eta|}{|Q|^2}$ and $-\frac{|Q|^2}{|\omega|^2}$ respectively, so that the assertion follows from the multiplicativity of $A_{\mathbb{E},\rho,\mathbb{F}Q}^{t^2,0}$ and $t : A_{\mathbb{E},\rho,\mathbb{F}Q}^{t^2,0} \rightarrow \mathbb{F}$ and the fact that the former vector norm was seen to be $|\omega|^2 D(\eta,\omega)$ in the proof of Lemma 9.5. For isotropic ω , we use the polynomial method from the proof of Lemma 9.5 again, and consider $(1 + \eta(\omega + s\xi))Q(1 + \eta(\omega + s\xi))^\rho$ and $D(\eta,\omega + s\xi)Q$ as A -valued polynomials in s , for some fixed, anisotropic $\xi \in (A_{\mathbb{E}}^-)_{\rho,\tilde{Q}}$. This means that we consider A as a vector space over \mathbb{F} , we choose a basis for it which includes Q , and consider the two sets of 16 polynomials in s arising as the coefficients using this basis (in the latter set, 15 polynomials will identically vanish and one is the coefficient $D(\eta,\omega + s\xi)$). By what we have proved, both sets of polynomials coincide for every s perhaps maybe $s = 0$ and $s = -\frac{2\langle\omega,\xi\rangle}{|\xi|^2}$, and the same argument as in the proof of Lemma 9.5 shows that they coincide for every s . The reduced norm condition required for $A_{\mathbb{E},\rho,\mathbb{F}Q}^{t^2,0}$ is satisfied by Lemma 9.5 itself. Substituting $s = 0$ verifies our assertion also for isotropic ω , which completes the proof of the lemma. \square

Denote $GSp_{A_{\mathbb{E}}} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}_{\rho,Q}$ the set of those elements $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ of $GSp_{A_{\mathbb{E}}} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, whose multipliers are in \mathbb{F}^\times , and which arise, in terms of Corollary 9.4, from parameters v and α from $(A_{\mathbb{E}}^-)_{\rho,Q}$, β which lies in $(A_{\mathbb{E}}^-)_{\rho,\tilde{Q}}$, and where a is assumed to be in $A_{\mathbb{E},\rho,\mathbb{F}Q}^{t^2}$. This definition is independent of the choice of parameters, as one sees in the following

Lemma 11.4. *Let c , w , γ , and δ be a set of parameters for some element of $GSp_{A_{\mathbb{E}}} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}_{\rho,Q}$ as in Corollary 9.4. If $w \in (A_{\mathbb{E}}^-)_{\rho,Q}$ then $c \in A_{\mathbb{E},\rho,\mathbb{F}Q}^{t^2}$, $\gamma \in (A_{\mathbb{E}}^-)_{\rho,Q}$, and $\delta \in (A_{\mathbb{E}}^-)_{\rho,\tilde{Q}}$.*

Proof. By definition, there is a set $a \in A_{\mathbb{E},\rho,\mathbb{F}Q}^{t^2}$, $v \in (A_{\mathbb{E}}^-)_{\rho,Q}$, $\alpha \in (A_{\mathbb{E}}^-)_{\rho,Q}$, and $\beta \in (A_{\mathbb{E}}^-)_{\rho,\tilde{Q}}$ of parameters from Corollary 9.4 for that element. Given w as an alternative parameter, the other parameters c , γ , and δ are determined by the formulae from Lemma 9.6. The assumptions on a , w , v , and β and Lemma 11.3 shows that the two multipliers in the expression for c in part (i) of that lemma belong to $A_{\mathbb{E},\rho,\mathbb{F}Q}^{t^2}$. Part (i) of Lemma 11.2 shows that the expression for δ in part (ii) of Lemma 9.6 is in $(A_{\mathbb{E}}^-)_{\rho,\tilde{Q}}$, as that the denominator $D(\beta,w - v)$ was seen to lie in \mathbb{F}^\times . When we examine the expression for γ in part (iii) of Lemma

9.6, we get the image of the action of a^{-1} on some vector, which lies in $(A_{\mathbb{E}}^-)_{\rho,Q}$ by part (iv) of Lemma 11.2 and the \mathbb{F} -rationality of m and of the denominator $D(\beta, w - v)$ again. Since a , hence also a^{-1} , comes from $A_{\mathbb{E},\rho,\mathbb{F}Q}^{t^2}$, this expression also lies in $(A_{\mathbb{E}}^-)_{\rho,Q}$ by Lemma 7.4. This completes the proof of the lemma. \square

We can now prove

Proposition 11.5. *The set $GSp_{A_{\mathbb{E}}} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}_{\rho,Q}$ is a subgroup of $GSp_{A_{\mathbb{E}}} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, which is stable under ρ . It is contained in $GSp_{A_{\mathbb{E}}}^{\mathbb{E}^2} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ and comes endowed with a splitting map into the double cover $\widetilde{GSp}_{A_{\mathbb{E}}}^{\mathbb{E}^2} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$.*

Proof. As in the proofs of Proposition 9.8 and Theorem 9.10, given two elements of $GSp_{A_{\mathbb{E}}} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}_{\rho,Q}$, we may assume that in the form of Corollary 9.4, the left multiplier g has parameters a, v, α and β , the right multiplier h arises from e, z, κ , and ν , and x , the same v, ξ , and ζ are parameters for the product gh . We may further assume that v and z lie in $(A_{\mathbb{E}}^-)_{\rho,Q}$. Lemma 11.4 implies that α and κ also come from the same space, β and ν are in $(A_{\mathbb{E}}^-)_{\rho,\tilde{Q}}$, and a and e are elements of $A_{\mathbb{E},\rho,\mathbb{F}Q}^{t^2}$. We have to show that the remaining parameters for gh lie in the appropriate sets. For x this follows from the properties of a, e, α, z , and ν by Lemma 11.3. Invoking part (iv) of Lemma 11.2, as well as Lemma 7.4 for the action of $e^{-1} \in A_{\mathbb{E},\rho,\mathbb{F}Q}^{t^2}$, verifies the assertion for ξ , since the multiplier n and the denominator $D(\alpha + z, \nu)$ lie in \mathbb{F}^\times by the proof of Lemma 11.3. Now, ζ is the sum of β and the image of the action of \bar{a}^{-1} on a vector, which is contained in $(A_{\mathbb{E}}^-)_{\rho,\tilde{Q}}$ by part (i) of Lemma 11.2. By parts (i) and (iv) of the latter lemma, it is sufficient to show that $\tilde{\zeta} - \tilde{\beta} \in (A_{\mathbb{E}}^-)_{\rho,Q}$. But using Lemma 5.6 and part (i) of Lemma 11.2 again, the latter vector is obtained by an element of $(A_{\mathbb{E}}^-)_{\rho,Q}$ by the action of $a \in A_{\mathbb{E},\rho,\mathbb{F}Q}^{t^2}$ (up to scalars from \mathbb{F}^\times), which verifies the assertion for ζ as well. Hence $GSp_{A_{\mathbb{E}}} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}_{\rho,Q}$ is a subgroup of $GSp_{A_{\mathbb{E}}} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$. For the stability under ρ , we may write any element of $GSp_{A_{\mathbb{E}}} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}_{\rho,Q}$ as in Corollary 9.4, with parameters as in Lemma 11.4. The fact that ρ preserves $(A_{\mathbb{E}}^-)_{\rho,Q}$, $(A_{\mathbb{E}}^-)_{\rho,\tilde{Q}}$, and $A_{\mathbb{E},\rho,\mathbb{F}Q}^{t^2}$ implies the preservation of $GSp_{A_{\mathbb{E}}} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}_{\rho,Q}$ as well. Now, the value of the map φ from Proposition 9.8 is based only on the parameter from $A_{\mathbb{E}}^\times$ in Corollary 9.4. As for matrices in $GSp_{A_{\mathbb{E}}} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}_{\rho,Q}$ these parameters come from $A_{\mathbb{E},\rho,\mathbb{F}Q}^{t^2}$ and the latter group is contained in $A_{\mathbb{E}}^{(\mathbb{E}^\times)^2}$ by the definition of the former group in Lemma 7.2, our subgroup is contained in $GSp_{A_{\mathbb{E}}}^{\mathbb{E}^2} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$. In addition, the double cover $\widetilde{GSp}_{A_{\mathbb{E}}}^{\mathbb{E}^2} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ is defined by adding a choice of a square root for the reduced norm of the parameter from $A_{\mathbb{E}}^{(\mathbb{E}^\times)^2}$, so that we get a parameter from $\widetilde{A}_{\mathbb{E}}^{(\mathbb{E}^\times)^2}$ for elements of this double cover. But $\widetilde{A}_{\mathbb{E}}^{(\mathbb{E}^\times)^2}$ was seen to split over $A_{\mathbb{E},\rho,\mathbb{F}Q}^{t^2}$ via $g \mapsto (g, t(g))$. Moreover, this splitting map is compatible with parameter changes inside $A_{\mathbb{E},\rho,\mathbb{F}Q}^{t^2}$, $(A_{\mathbb{E}}^-)_{\rho,Q}$, and $(A_{\mathbb{E}}^-)_{\rho,\tilde{Q}}$ by part (i) of

Lemma 9.6 and Lemma 11.3. This establishes the splitting of $\widetilde{GSp}_{A_{\mathbb{E}}}^{\mathbb{E}^2} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ over $GSp_{A_{\mathbb{E}}} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}_{\rho, Q}$ as well, and completes the proof of the proposition. \square

In case we wish to evaluate ψ -images of elements of $GSp_{A_{\mathbb{E}}} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}_{\rho, Q}$ using other entries, we cannot use the matrix $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, as it does not belong to this group. However, the element $\begin{pmatrix} 0 & Q \\ \tilde{Q} & 0 \end{pmatrix}$, of multiplier $-|Q|^2$, does belong there: By choosing some non-zero $h \in \mathbb{E}_0$, we recall that hQ and $\frac{Q}{h|Q|^2}$ are in $(A_{\mathbb{E}}^-)_{\rho, Q}$, $\frac{\tilde{Q}}{h|Q|^2} \in (A_{\mathbb{E}}^-)_{\rho, Q}$, and $h|Q|^2$ is in $A_{\mathbb{E}, \rho, \mathbb{F}Q}^{\mathbb{E}^2}$, and our element may be obtained from the parameters $v = -hQ$, $a = h|Q|^2$, $\alpha = \frac{Q}{h|Q|^2}$, and $\beta = \frac{\tilde{Q}}{h|Q|^2}$. Recall that the multiplier $t(h|Q|^2)$ of $h|Q|^2 \in A_{\mathbb{E}, \rho, \mathbb{F}Q}^{\mathbb{E}^2}$ is $-h^2|Q|^4$, so that as an element of $GSp_{A_{\mathbb{E}}} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}_{\rho, Q}$ we must have $\psi \begin{pmatrix} 0 & Q \\ \tilde{Q} & 0 \end{pmatrix} = \begin{pmatrix} 0 & -\tilde{Q} \\ -Q & 0 \end{pmatrix}$. In any case, we may use this element in order to transfer the formula for ψ on $GSp_{A_{\mathbb{E}}} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}_{\rho, Q}$ to be based on the other matrix entries.

As in the situation we encountered in dimension 6 and general discriminant, we remark that unless Q^2 is a scalar and Q and \tilde{Q} span the same vector space, the space $(A_{\mathbb{E}}^-)_{\rho, Q} \oplus H$ is not invariant under the linear automorphism $\hat{\psi}$ from Lemma 9.11. In addition, the automorphism ψ does now preserve the subgroup $GSp_{A_{\mathbb{E}}} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}_{\rho, Q}$ (embedded via the splitting map from Proposition 11.5) in this case. However, we can still pursue the usual route by using the following

Lemma 11.6. *Let \hat{Q} denote the element of $\widetilde{GSp}_{A_{\mathbb{E}}}^{\mathbb{E}^2} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ lying over $\begin{pmatrix} Q & 0 \\ 0 & -\tilde{Q} \end{pmatrix}$ in which the square root t of $N_{\mathbb{E}}^{A_{\mathbb{E}}}(Q)$ is chosen to be $|Q|^2$. Then the element $\hat{Q}\tilde{\psi}$ of the semi-direct product of $\{1, \tilde{\psi}\}$ and $\widetilde{GSp}_{A_{\mathbb{E}}}^{\mathbb{E}^2} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ from Lemma 9.14 squares to $-|Q|^2$ (with $|Q|^4$ as the square root of its reduced norm) in that semi-direct product. Its action on $(A_{\mathbb{E}}^-)_{\rho, Q} \oplus H$ coincides with that of ρ (which preserves it), and conjugation by this element operates on $\widetilde{GSp}_{A_{\mathbb{E}}}^{\mathbb{E}^2} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ via $g \mapsto \hat{Q}\psi(g)\hat{Q}^{-1}$. This automorphism of $\widetilde{GSp}_{A_{\mathbb{E}}}^{\mathbb{E}^2} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ has order 2, and it preserves the subgroup $GSp_{A_{\mathbb{E}}} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}_{\rho, Q}$ embedded through the splitting map from Proposition 11.5 as its operation on the latter group is the same as that of ρ .*

Proof. The multiplier of \hat{Q} is $|Q|^2$, and as $Q \in A_{\mathbb{E}}^-$ and $t = |Q|^2$ indeed satisfies $t^2 = N_{\mathbb{F}}^A(Q)$ by Corollary 5.4, the definition of the map ψ in Theorem 9.10 shows that $\psi(\hat{Q})$ is $\begin{pmatrix} -\tilde{Q} & 0 \\ 0 & Q \end{pmatrix}$ (with the same square root $-|Q|^2$). The first assertion follows immediately from the fact that $\hat{Q}\psi(\hat{Q}) = -|Q|^2 I$ and the product of the square roots is $|Q|^4$ (the number $D(\alpha + z, w)$ appearing in the proof of Theorem 9.10 in the square root of the reduced norm of x from Lemma 9.7 is just 1). Now, $\tilde{\psi}$ operates as $\hat{\psi}$ on $A_{\mathbb{E}}^- \oplus H_{\mathbb{E}}$ (which is just θ on the $A_{\mathbb{E}}^-$ part), and the operation of the diagonal element \hat{Q} was evaluated in Proposition 9.12: The $H_{\mathbb{E}}$ part is pointwise fixed since $t = m$ for our element, and the combined operation on $A_{\mathbb{E}}^-$ is via $u \mapsto -\frac{Q\tilde{u}Q}{|Q|^2}$. But Lemma 7.1 shows that

on $(A_{\mathbb{E}}^-)_{\rho,Q}$ the latter map coincides with ρ , and as ρ leaves H also pointwise fixed, this establishes the second assertion. The formula for the conjugation on $\widetilde{GSp}_{A_{\mathbb{E}}}^{\mathbb{E}^2} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ follows directly from the structure of the semi-direct product in Lemma 9.14, and it is of order 2 either since $-|Q|^2 I$ (with the square root $|Q|^4$ of $N_{\mathbb{E}}^{A_{\mathbb{E}}}(-|Q|^2)$) operates trivially or by a direct evaluation. In order to examine the action of this automorphism on $GSp_{A_{\mathbb{E}}} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}_{\rho,Q}$ in view of that of ρ , we recall the relation between \hat{Q} and $\psi(\hat{Q})$, so that we evaluate $-\frac{1}{|Q|^2} \hat{Q} g \psi(\hat{Q})$ for $g \in GSp_{A_{\mathbb{E}}} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}_{\rho,Q}$ and compare it with g^ρ . It suffices to take g from a set of generators of the latter group, namely $\begin{pmatrix} 1 & v \\ 0 & 1 \end{pmatrix}$ with $v \in (A_{\mathbb{E}}^-)_{\rho,Q}$, $\begin{pmatrix} 1 & 0 \\ w & 1 \end{pmatrix}$ with $w \in (A_{\mathbb{E}}^-)_{\rho,\hat{Q}}$, and $\begin{pmatrix} a & 0 \\ 0 & m\bar{a}^{-1} \end{pmatrix}$ where a lies in $A_{\mathbb{E},\rho,\mathbb{F}Q}^{t^2}$. Now, ψ replaces v and w by their θ -images and a by $t(a)\bar{a}^{-1}$, and after conjugating by \hat{Q} and using Lemma 5.2 we find that v is sent to $-\frac{Q\bar{v}Q}{|Q|^2}$, w is taken to $-\frac{Q\bar{w}Q}{|Q|^2}$, the diagonal entries of the unipotent generators remain invariant, and a is mapped to $t(a)Q\bar{a}^{-1}Q^{-1}$ (the other entry \bar{a}^{-1} becomes $\frac{Q^{-1}aQ}{t(a)}$). But as we assume that $v \in (A_{\mathbb{E}}^-)_{\rho,Q}$, $w \in (A_{\mathbb{E}}^-)_{\rho,\hat{Q}}$, and $a \in A_{\mathbb{E},\rho,\mathbb{F}Q}^{t^2}$, the first two expressions are v^ρ and w^ρ by Lemma 7.1, while the proof of Lemma 7.3 shows that the latter expression is just a^ρ (and the one in parentheses is $\bar{a}^{-\rho}$). Since the multiplier m lies in \mathbb{F} and is thus ρ -invariant, this establishes the coincidence of ρ and conjugation by $\hat{Q}\tilde{\psi}$ on $GSp_{A_{\mathbb{E}}} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}_{\rho,Q}$, and as ρ was seen to preserve this group in Proposition 11.5, conjugation by $\hat{Q}\tilde{\psi}$ does the same. This completes the proof of the lemma. \square

Note that $\Xi = \hat{Q}$ satisfies the conditions of Corollary 9.18, and the operations of $\hat{Q}\tilde{\psi}$ appearing in Lemma 11.6 are just $\hat{\psi}_{\hat{Q}}$ and $\psi_{\hat{Q}}$ respectively. Now, Lemma 11.6 gives an intrinsic description of $GSp_{A_{\mathbb{E}}} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}_{\rho,Q}$, and we can also establish some properties of its entries and the elements from the GSp relations:

Corollary 11.7. *The subgroup $GSp_{A_{\mathbb{E}}} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}_{\rho,Q}$ is characterized as the set of elements of $\widetilde{GSp}_{A_{\mathbb{E}}}^{\mathbb{E}^2} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ on which the automorphism $\psi_{\hat{Q}}$ operates as ρ .*

Proof. The fact that $\psi_{\hat{Q}}(g) = g^\rho$ for $g \in GSp_{A_{\mathbb{E}}} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}_{\rho,Q}$ was seen in the proof of Lemma 11.6. Conversely, let $g \in \widetilde{GSp}_{A_{\mathbb{E}}}^{\mathbb{E}^2} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ be an element satisfying $\psi_{\hat{Q}}(g) = g^\rho$. Then the multipliers of both sides coincide, and as ψ commutes with m and $m(g^\rho) = m(g)^\rho$ we find that $m(g) \in \mathbb{F}^\times$. Moreover, Lemma 9.3 allows us to find a set of parameters for Corollary 9.4 for g in which $v \in (A_{\mathbb{E}}^-)_{\rho,Q}$. As $\begin{pmatrix} 1 & v \\ 0 & 1 \end{pmatrix}$ lies in $GSp_{A_{\mathbb{E}}} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}_{\rho,Q}$, it suffices to consider elements $g \in \widetilde{GSp}_{A_{\mathbb{E}}}^{\mathbb{E}^2} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ having invertible upper left entry. But these elements have a unique decomposition as in Lemma 9.2. Comparing these decompositions for g^ρ and $\psi_{\hat{Q}}(g)$ and recalling that the unipotent matrices are assumed to have unipotent ψ -images (and not with -1 on the diagonal) reduces the verification to the multipliers appearing in Lemma 9.2, lifted into $\widetilde{GSp}_{A_{\mathbb{E}}}^{\mathbb{E}^2} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$. But these verifications are carried out in the proof of Lemma 11.6. This proves the corollary. \square

One can also show that if $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is an element of $GS_{p_{A_{\mathbb{E}}}}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}_{\rho, Q}$ then a , bQ^{-1} , Qc , and QdQ^{-1} all lie in $A_{\mathbb{E}, \rho, \mathbb{F}Q}^{t^2, 0}$, the elements $\bar{a}b = -b\bar{a}$ and $\bar{b}d = -d\bar{b}$ of $A_{\mathbb{E}}^-$ belong to $(A_{\mathbb{E}}^-)_{\rho, Q}$, and $\bar{a}c = -c\bar{a}$ and $\bar{c}d = -d\bar{c}$ come from $(A_{\mathbb{E}}^-)_{\rho, \tilde{Q}}$. In fact, conjugating by $\begin{pmatrix} 1 & 0 \\ 0 & Q \end{pmatrix}$ as in Corollary 9.17 yields a subgroup $GS_{p_{A_{\mathbb{E}}}}\begin{pmatrix} Q & 0 \\ 0 & Q \end{pmatrix}_{\rho, Q}$ of $GS_{p_{A_{\mathbb{E}}}}\begin{pmatrix} Q & 0 \\ 0 & Q \end{pmatrix}$ with a simpler description: All the entries of elements $\begin{pmatrix} e & f \\ g & h \end{pmatrix}$ of that group come from $A_{\mathbb{E}, \rho, \mathbb{F}Q}^{t^2, 0}$, and for which the elements $eQ\bar{f} = fQ\bar{e}$ and $gQ\bar{h} = hQ\bar{g}$ of $A_{\mathbb{E}}^-$ lie in $(A_{\mathbb{E}}^-)_{\rho, Q}$, while $\bar{e}Q^{-1}g = \bar{g}Q^{-1}e$ and $\bar{f}Q^{-1}h = \bar{h}Q^{-1}f$ are in $(A_{\mathbb{E}}^-)_{\rho, \tilde{Q}}$ (use parts (ii) and (v) of Lemma 11.2 again). Moreover, when many of these terms are invertible, some of the latter assertions follow from one another—see Lemma 11.2 and the remarks following it. However, once non-zero entries which are not invertible are involved, there may be some matrices satisfying these conditions which do not belong to $GS_{p_{A_{\mathbb{E}}}}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}_{\rho, Q}$. Hence we content ourselves with the description appearing in Corollary 11.7.

The reason for considering $GS_{p_{A_{\mathbb{E}}}}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}_{\rho, Q}$ is given in the following

Lemma 11.8. *Consider the operation of $GS_{p_{A_{\mathbb{E}}}}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}_{\rho, Q}$, viewed as a subgroup of $\widetilde{GS_{p_{A_{\mathbb{E}}}}}^{\mathbb{E}^2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ via the lift from Proposition 11.5, on the \mathbb{E} -vector space $A_{\mathbb{E}}^- \oplus H_{\mathbb{E}}$. This action preserves the \mathbb{F} -subspace $(A_{\mathbb{E}}^-)_{\rho, Q} \oplus H$. The group generated by $GS_{p_{A_{\mathbb{E}}}}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}_{\rho, Q}$ and the element $\tilde{Q}\psi$ from Lemma 11.6 containing the latter group with index 2, and maps into $O((A_{\mathbb{E}}^-)_{\rho, Q} \oplus H)$ as well.*

Proof. As in Proposition 9.12, it suffices to prove the assertion for a generating subset of the subgroup. Corollary 9.4 and the definition of $GS_{p_{A_{\mathbb{E}}}}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}_{\rho, Q}$ show that the set consisting of unipotent matrices $\begin{pmatrix} 1 & v \\ 0 & 1 \end{pmatrix}$ with $v \in (A_{\mathbb{E}}^-)_{\rho, Q}$ and $\begin{pmatrix} 1 & 0 \\ w & 1 \end{pmatrix}$ where $w \in (A_{\mathbb{E}}^-)_{\rho, \tilde{Q}}$ together with the subgroup of diagonal matrices $\begin{pmatrix} a & 0 \\ 0 & m\bar{a}^{-1} \end{pmatrix}$ with $a \in A_{\mathbb{E}, \rho, \mathbb{F}Q}^{t^2}$ and $m \in \mathbb{F}^\times$ is such a generating set. The action of these generators on an arbitrary element $\begin{pmatrix} u & -p \\ q & -\bar{u} \end{pmatrix} \in A_{\mathbb{E}}^- \oplus H_{\mathbb{E}}$ was seen in the proof of Proposition 9.12 to be as follows: The u coordinate becomes $u + qv$, $u + p\tilde{w}$, and $\frac{au\bar{u}}{t(a)}$ (recall the choice of the ψ -image), p is sent to $p + 2\langle u, v \rangle + q|v|^2$, p , and $\frac{t(a)p}{m}$, and q is mapped to q , $q + 2\langle u, \tilde{w} \rangle + q|w|^2$, and $\frac{mq}{t(a)}$, respectively. Our assumptions on v , w , m , and a show, using part (iv) of Lemma 11.2 for w and Lemma 7.4 for a , that if p and q are from \mathbb{F} and $u \in (A_{\mathbb{E}}^-)_{\rho, Q}$ then the same assertion holds for their images. The remaining assertions follow, as in the proof of Lemma 7.4, from Lemma 11.6, Lemma 9.14 over \mathbb{E} , and the fact that $(A_{\mathbb{E}}^-)_{\rho, Q} \oplus H$ inherits its quadratic structure from $A_{\mathbb{E}}^- \oplus H_{\mathbb{E}}$. This proves the lemma. \square

The assertion about reflections in this case appears in the following

Lemma 11.9. *Let g be an anisotropic element of $(A_{\mathbb{E}}^-)_{\rho, Q} \oplus H$. Then the product $g\hat{Q}^{-1}$ belongs to $GS_{p_{A_{\mathbb{E}}}}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}_{\rho, Q}$, and if we compose the action of the*

element $\hat{Q}\tilde{\psi}$ from Lemma 11.6 with that of the latter composition we obtain the reflection in g on $(A_{\mathbb{E}}^-)_{\rho,Q} \oplus H$.

Proof. First, the fact that $(A_{\mathbb{E}}^-)_{\rho,Q} \oplus H$ is a quadratic space over \mathbb{F} means that all the multipliers are from \mathbb{F} . Consider the element $g\hat{Q}^{-1}$, for anisotropic $g = \begin{pmatrix} u & -p \\ q & -u \end{pmatrix} \in (A_{\mathbb{E}}^-)_{\rho,Q} \oplus H$. If $p \neq 0$, we multiply it by $\begin{pmatrix} 0 & 1 \\ \hat{Q} & 0 \end{pmatrix}$ from the right. As the product of $\hat{Q}^{-1} = \frac{\psi(\hat{Q})}{|Q|^2}$ and the latter element yields the matrix $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ from the proof of Lemma 9.11, we may use the parameters given in that lemma for this case. As the scalar $p \in \mathbb{F}^\times$ lies in $A_{\mathbb{E},\rho,\mathbb{F}Q}^{t^2}$, the vector $\frac{u}{p}$ is in $(A_{\mathbb{E}}^-)_{\rho,Q}$ by our assumption on U , and $\frac{\tilde{u}}{p} \in (A_{\mathbb{E}}^-)_{\rho,\tilde{Q}}$ by part (i) of Lemma 11.2, we indeed get $g\hat{Q}^{-1} \in GSp_{A_{\mathbb{E}}} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}_{\rho,Q}$ if $p \neq 0$. with $p = 0$, so that $u \in A_{\mathbb{E}}^\times$, we get a matrix of multiplier $\frac{|u|^2}{|Q|^2}$ for which the parameters may be taken to be $v = \alpha = 0$, $a = uQ^{-1}$, and $\beta = \frac{q\tilde{u}}{|u|^2}$. Parts (i) and (iii) of Lemma 11.2 show that $g\hat{Q}^{-1} \in GSp_{A_{\mathbb{E}}} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}_{\rho,Q}$ in case $p = 0$ as well. We must, however, consider these elements in the double cover $\widetilde{GSp}_{A_{\mathbb{E}}} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$. Recall from the proof of Lemma 9.11 that the branch of ψ which coincides with $\hat{\psi}$ is determined for $p \neq 0$ by the condition that after right multiplication by $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ (considered as its own ψ -image) the square root of $N_{\mathbb{F}}^A(p) = p^4$ is p^2 , while for $p = 0$ we take $-|u|^2$ for the square root of $N_{\mathbb{F}}^A(u)$. One verifies that the chosen ψ -image in the definition of \hat{Q} and \hat{Q}^{-1} in Lemma 11.6 and the ψ -image of $\begin{pmatrix} 0 & 1 \\ \hat{Q} & 0 \end{pmatrix}$ as an element of $GSp_{A_{\mathbb{E}}} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}_{\rho,Q}$ combine to give $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ with itself as its ψ -image, and indeed p^2 is the multiplier of $p \in A_{\mathbb{E},\rho,\mathbb{F}Q}^{t^2}$ as a parameter of $g \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$. On the other hand, part (iii) of Lemma 11.2 shows that the multiplier of the parameter uQ^{-1} is $-\frac{|u|^2}{|Q|^2}$, and multiplying it by the chosen square root $|Q|^2$ of the diagonal element \hat{Q} yields the desired value $-|u|^2$. It follows that the composition of $g\hat{Q}^{-1}$ and $\hat{Q}\tilde{\psi}$ is the combination $g\tilde{\psi}$, with $g \in \widetilde{GSp}_{A_{\mathbb{E}}} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ whose ψ -image coincides with $\hat{\psi}(g)$. The action of this element on $A_{\mathbb{E}}^- \oplus H_{\mathbb{E}}$ was seen in Lemma 9.15 (over \mathbb{E}) to be the reflection in g , and this assertion descends to $(A_{\mathbb{E}}^-)_{\rho,Q} \oplus H$ for anisotropic g which is taken from the latter space. This completes the proof of lemma. \square

Observe that Lemma 11.9 in fact uses the alternative representation appearing in Corollary 9.18 with $\Xi = \hat{Q}$, with right multiplication by \hat{Q}^{-1} on the space and $\psi_{\hat{Q}}$ as the automorphism of the group.

Now we are in a position to prove

Theorem 11.10. *We have $Gspin((A_{\mathbb{E}}^-)_{\rho,Q} \oplus H) = GSp_{A_{\mathbb{E}}} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}_{\rho,Q}$, and the spin group $spin((A_{\mathbb{E}}^-)_{\rho,Q} \oplus H)$ is the subgroup $Sp_{A_{\mathbb{E}}} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}_{\rho,Q}$ consisting of those elements of $GSp_{A_{\mathbb{E}}} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}_{\rho,Q}$ having multiplier 1 (namely the intersection of $GSp_{A_{\mathbb{E}}} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}_{\rho,Q}$ with $\widetilde{Sp}_{A_{\mathbb{E}}} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$).*

Proof. We have a surjective map from the semi-direct product of $\{1, \tilde{\psi}\}$ with $\widetilde{GSp}_{A_{\mathbb{E}}} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ (which we may consider as generated by $\hat{Q}\tilde{\psi}$ and the latter group) onto $O(A_{\mathbb{E}}^- \oplus H_{\mathbb{E}})$, whose kernel is \mathbb{E}^\times , and such that the inverse image of $SO(A_{\mathbb{E}}^- \oplus H_{\mathbb{E}})$ is $\widetilde{GSp}_{A_{\mathbb{E}}} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$. By Lemma 11.8, the subgroup which $\hat{Q}\tilde{\psi}$ generates with $GSp_{A_{\mathbb{E}}} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}_{\rho, Q}$ is sent to the subgroup $O((A_{\mathbb{E}}^-)_{\rho, Q} \oplus H)$ of $O(A_{\mathbb{E}}^- \oplus H_{\mathbb{E}})$. Invoking Lemma 11.9 and Proposition 2.2, we obtain that this subgroup surjects onto $O((A_{\mathbb{E}}^-)_{\rho, Q} \oplus H)$. By Theorem 9.16 and the fact that the determinant commutes with the injection of $O((A_{\mathbb{E}}^-)_{\rho, Q} \oplus H)$ into $O(A_{\mathbb{E}}^- \oplus H_{\mathbb{E}})$, we find that an element of the group generated by $\hat{Q}\tilde{\psi}$ and $GSp_{A_{\mathbb{E}}} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}_{\rho, Q}$ is sent to $SO((A_{\mathbb{E}}^-)_{\rho, Q} \oplus H)$ if and only if it comes from $GSp_{A_{\mathbb{E}}} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}_{\rho, Q}$ alone. It follows that the map $GSp_{A_{\mathbb{E}}} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}_{\rho, Q} \rightarrow O((A_{\mathbb{E}}^-)_{\rho, Q} \oplus H)$ is surjective. An element $GSp_{A_{\mathbb{E}}} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}_{\rho, Q}$ acts trivially if and only if it is a scalar matrix rI (with $r \in \mathbb{E}^\times$) such that its ρ -image is the same as $\hat{Q}\psi(rI)\hat{Q}^{-1}$, and moreover we require $\psi(rI)$ to be $+rI$ in the double cover $\widetilde{GSp}_{A_{\mathbb{E}}} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$. This happens if and only if $r \in \mathbb{F}^\times$ (non-zero elements $r \in \mathbb{E}_0$ satisfy the first condition but have the wrong sign of $\psi(rI)$, hence they do not operate trivially but rather as $-Id_{(A_{\mathbb{E}}^-)_{\rho, Q} \oplus H}$). This proves that $Gspin((A_{\mathbb{E}}^-)_{\rho, Q} \oplus H) = GSp_{A_{\mathbb{E}}} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}_{\rho, Q}$. The spinor norm of an element of $\widetilde{GSp}_{A_{\mathbb{E}}} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ was seen in Theorem 9.16 to be the image of its multiplier m , so that the same assertion holds for the images of $GSp_{A_{\mathbb{E}}} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}_{\rho, Q}$ in $SO((A_{\mathbb{E}}^-)_{\rho, Q} \oplus H)$. It follows that $SO^1((A_{\mathbb{E}}^-)_{\rho, Q} \oplus H)$ consists of images of those $g \in GSp_{A_{\mathbb{E}}} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}_{\rho, Q}$ such that $m(g) \in (\mathbb{F}^\times)^2$. Dividing by scalars, we restrict attention to those g with $m(g) = 1$, i.e., to $g \in Sp_{A_{\mathbb{E}}} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}_{\rho, Q}$, and as the kernel of the restriction to $Sp_{A_{\mathbb{E}}} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}_{\rho, Q}$ is $\{\pm 1\}$, the latter group is indeed the $spin((A_{\mathbb{E}}^-)_{\rho, Q} \oplus H)$. This completes the proof of the theorem. \square

When we wish to consider the independence of our groups of the choices which we made, we first observe that by the Witt Cancellation Theorem, the complement of any hyperbolic plane inside an isotropic space is independent (up to isomorphism) of the specific hyperbolic plane we took. Hence it remains to see what happens when we change the choices for the complement $(A_{\mathbb{E}}^-)_{\rho, Q}$. For this we again denote by σ the map on $A_{\mathbb{E}}$ which was previously denoted ρ , and let Q and σ vary. The resulting independence assertion appears in

Proposition 11.11. *Let σ, τ , and η be ring automorphisms of $A_{\mathbb{E}}$, all of order 2, which restrict to ρ on \mathbb{E} , and let Q, R, S and T be elements of $A_{\mathbb{E}}^-$ which satisfy the conditions of parts (i), (ii), and (iii) of Proposition 7.7. Assume further that the element $T = Qb^{-1}$ of $A_{\mathbb{E}}^-$ has vector norm $\frac{|Q|^2}{bb}$. Then the four groups $GSp_{A_{\mathbb{E}}} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}_{\sigma, Q}$, $GSp_{A_{\mathbb{E}}} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}_{\sigma, R}$, $GSp_{A_{\mathbb{E}}} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}_{\tau, S}$, and $GSp_{A_{\mathbb{E}}} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}_{\eta, T}$ are isomorphic in such a way that the isomorphisms take the Sp subgroups to one another.*

Proof. The proof of Proposition 7.7 shows that $A_{\mathbb{E},\sigma,\mathbb{F}Q}^{t^2}$ coincides with $A_{\mathbb{E},\eta,\mathbb{F}S}^{t^2}$, and conjugation by e takes this group to $A_{\mathbb{E},\tau,\mathbb{F}eQ\bar{e}}^{t^2}$. Moreover, this proof yields the existence of two elements c and d of $A_{\mathbb{E}}^\times$, with $c^\sigma = c$ and $d^\tau = d$, such that c conjugates $A_{\mathbb{E},\sigma,\mathbb{F}Q}^{t^2}$ to $A_{\mathbb{E},\sigma,\mathbb{F}R}^{t^2}$ and d conjugates $A_{\mathbb{E},\tau,\mathbb{F}eQ\bar{e}}^{t^2}$ to $A_{\mathbb{E},\tau,\mathbb{F}S}^{t^2}$. We first claim that the action of c (resp. e) on $A_{\mathbb{E}}^-$ sends the space $(A_{\mathbb{E}}^-)_{\sigma,Q}$ to $(A_{\mathbb{E}}^-)_{\sigma,R}$ (resp. $(A_{\mathbb{E}}^-)_{\tau,eQ\bar{e}}$). Indeed, for any $v \in A_{\mathbb{E}}^-$ we have $(cv\bar{c})^\sigma = cv^\sigma\bar{c}$ and $(ev\bar{e})^\tau = ev^\tau\bar{e}$ (this is clear for the σ -invariant element c , and for e we use the fact that $x^\tau = ee^{-\sigma}x^\sigma e^\sigma e^{-1}$ and the invariance of $ee^{-\sigma}$ under $x \mapsto \bar{x}^\sigma$). Hence if $v \in (A_{\mathbb{E}}^-)_{\sigma,Q}$ then we substitute the value of v^σ from Lemma 7.1, and then using Lemma 5.6 and Proposition 5.5 we get the asserted result (recall that $R = rcQ\bar{c}$). It follows that the operation of d maps $(A_{\mathbb{E}}^-)_{\tau,eQ\bar{e}}$ to $(A_{\mathbb{E}}^-)_{\tau,S}$. Next, we show that $(A_{\mathbb{E}}^-)_{\eta,T} = (A_{\mathbb{E}}^-)_{\sigma,Q}$ as well. To see this, first observe that if $\bar{b}^\sigma = b$ and $b\bar{b} \in \mathbb{E}^\times$ then applying σ to the latter scalar yields $\bar{b}b$. As b is invertible (with inverse $\frac{\bar{b}}{b\bar{b}}$), this shows that the latter scalar lies in fact in \mathbb{F}^\times . The fact that both Q and $T = Qb^{-1}$ are in $A_{\mathbb{E}}^-$ allows us now to write T also as $\frac{bQ}{b\bar{b}}$. Hence if $v \in (A_{\mathbb{E}}^-)_{\sigma,Q}$ then using the definition of η and Lemma 7.1 we find that $v^\eta = -b\bar{b}\frac{T\bar{v}T}{|Q|^2}$, which equals $-\frac{T\bar{v}T}{|T|^2}$ by our assumption on $|T|^2$. This proves that v indeed lies in $(A_{\mathbb{E}}^-)_{\eta,T}$. The equality of the two spaces, as well as the fact that the entire spaces with R , $eQ\bar{e}$, and S are in the image of $(A_{\mathbb{E}}^-)_{\sigma,Q}$, follows either by inverting the above argument or by comparing dimensions (and using the injectivity of all the operations considered here).

Let now g be an element of $GSp_{A_{\mathbb{E}}} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}_{\sigma,Q}$, and we present it by using the appropriate parameters in Corollary 9.4. The previous paragraph shows that by using the same parameters we also get that $g \in GSp_{A_{\mathbb{E}}} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}_{\eta,T}$. For the other groups, observe that when we conjugate the generators $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, $\begin{pmatrix} 1 & 0 \\ w & 1 \end{pmatrix}$, and $\begin{pmatrix} a & 0 \\ 0 & m\bar{a}-1 \end{pmatrix}$ of the group $GSp_{A_{\mathbb{E}}} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ by some diagonal element of the form $\begin{pmatrix} x & 0 \\ 0 & \bar{x}-1 \end{pmatrix}$ (of multiplier 1), then v , w , and a are taken to $xv\bar{x}$, $\bar{x}^{-1}wx^{-1}$, and $xa\bar{x}^{-1}$ respectively. Moreover, if $w = \tilde{u}$ for some $u \in A_{\mathbb{E}}^-$ then its image is $\widehat{xu\bar{x}}$ (up to the scalar $N_{\mathbb{F}}^A(x)$) by Lemma 5.6. For the generators of $GSp_{A_{\mathbb{E}}} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}_{\sigma,Q}$, in which $v \in (A_{\mathbb{E}}^-)_{\sigma,Q}$, $w \in (A_{\mathbb{E}}^-)_{\sigma,\bar{Q}}$, and $a \in A_{\mathbb{E},\sigma,\mathbb{F}Q}^{t^2}$, we have seen that for $x = c$ (resp. $x = e$) the images of v and a lie in $(A_{\mathbb{E}}^-)_{\sigma,R}$ and $A_{\mathbb{E},\sigma,\mathbb{F}Q}^{t^2}$ (resp. $(A_{\mathbb{E}}^-)_{\tau,eQ\bar{e}}$ and $A_{\mathbb{E},\tau,\mathbb{F}eQ\bar{e}}^{t^2}$) respectively. Moreover, $w = \tilde{u}$ for $u \in (A_{\mathbb{E}}^-)_{\sigma,Q}$ by parts (i) and (iv) of Lemma 11.2, and the fact that both c and e have reduced norms in \mathbb{F} shows that its image is the θ -image of an element which lies in $(A_{\mathbb{E}}^-)_{\sigma,R}$ (resp. $(A_{\mathbb{E}}^-)_{\tau,eQ\bar{e}}$). Hence part (i) of Lemma 11.2 proves the assertion for w as well. This shows that conjugation by $\begin{pmatrix} c & 0 \\ 0 & \bar{c}-1 \end{pmatrix}$ (resp. $\begin{pmatrix} e & 0 \\ 0 & \bar{e}-1 \end{pmatrix}$) sends $GSp_{A_{\mathbb{E}}} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}_{\sigma,Q}$ to $GSp_{A_{\mathbb{E}}} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}_{\sigma,R}$ (resp. $GSp_{A_{\mathbb{E}}} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}_{\tau,eQ\bar{e}}$), and further conjugation of the latter group by $\begin{pmatrix} d & 0 \\ 0 & \bar{d}-1 \end{pmatrix}$ takes the latter group to $GSp_{A_{\mathbb{E}}} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}_{\tau,S}$. The fact that these identifications and conjugations yields the full groups is established either by inverting the above argument, or by observing that since the maps and identifications from the previous paragraph are all surjective, we get all the

generators of the required groups in this process. Since our identifications and isomorphisms commute with the multiplier maps to \mathbb{F}^\times , the Sp groups are sent to one another in this process. This completes the proof of the proposition. \square

Note that as both η and $v \mapsto -\frac{T\bar{v}T^2}{|T|}$ are involutions which separate $A_{\mathbb{E}}^-$ to ± 1 -eigenspaces, the proof of Proposition 11.11 already shows that the vector norm of $T = Qb^{-1}$ must be $\pm \frac{|Q|^2}{bb}$. It seems likely that only the $+$ sign is possible (making the additional assumption in Proposition 11.11 redundant), but we have not checked this out in detail. The remarks about the transitivity of the relations from Proposition 7.7 on the possible choices of ring automorphisms of $A_{\mathbb{E}}^-$ (commuting with $\iota_B \otimes \iota_C$ and reducing to ρ on \mathbb{E} as usual) extend to similar assertions for Proposition 11.11.

Going back to our previous notation, with ρ on $A_{\mathbb{E}}$ as well as on $M_2(A_{\mathbb{E}})$, the spaces with more isotropic vectors which appear in this case are considered in the following

Corollary 11.12. *Assume that after extending scalars to \mathbb{E} , the quadratic space $((A_{\mathbb{E}}^-)_{\rho, Q} \oplus H)_{\mathbb{E}} = A_{\mathbb{E}}^- \oplus H_{\mathbb{E}}$ splits more than one hyperbolic plane. Then there exists some quaternion algebra B over \mathbb{F} and some number $\delta \in \mathbb{F}^\times$ representing a class modulo $N_{\mathbb{F}}^B(B^\times)$ such that the following assertions hold: The quadratic space is $\mathbb{E} \oplus B \oplus H$ with the norms from \mathbb{E} multiplied by $-\delta$, and its $Gspin$ and spin groups, denoted $GSp_4(B_{\mathbb{E}})_{\rho, \delta}$ and $Sp_4(B_{\mathbb{E}})_{\rho, \delta}$ respectively, consist of those elements of $GSp_4(B_{\mathbb{E}})$ and $Sp_4(B_{\mathbb{E}})$ on which ψ is defined as in the remark following Corollary 9.17, and conjugating it by the diagonal matrix with diagonal entries $-\delta, 1, -1$, and δ operates in the same way as ρ . When the latter \mathbb{E} -space splits more than two hyperbolic planes, meaning that it is the direct sum of 4 hyperbolic planes, then there exists representatives ε of a class modulo $N_{\mathbb{F}}^{\mathbb{E}}(\mathbb{E}^\times)$ and δ of a class with respect to $N_{\mathbb{F}}^B(B^\times)$ for $B = (\mathbb{E}, \rho, \varepsilon)$, for which the following holds: The space is the direct sum of a hyperbolic plane and three copies of \mathbb{E} , with the norms in two copies are multiplied by $-\varepsilon$ and $-\delta$, and the spin group is a the subgroup of a double cover of $SO^1\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$ over \mathbb{E} on which ρ coincides with the conjugation of ψ by the diagonal 8×8 matrix whose diagonal entries are $\delta\varepsilon, -\delta, -\varepsilon, 1, \varepsilon, -1, -\delta\varepsilon$, and δ . The $Gspin$ group is a the subgroup of the spinor norm related subgroup of the general special orthogonal group of $\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$ which is defined by the same relation between ρ and ψ .*

Proof. The existence of B and δ , as well as ε , is a consequence of Corollary 7.8, from which we also adopt the choice of Q to be $\begin{pmatrix} 0 & \delta \\ 1 & 0 \end{pmatrix}$. The form of the spaces is then given in Corollary 7.9. The description of the $Gspin$ and spin groups now follows from Theorem 11.10, Corollary 11.7, and Corollary 9.17, taking into consideration the conjugation by $\begin{pmatrix} 1 & 0 \\ 0 & R \end{pmatrix}$ or $\begin{pmatrix} 1 & 0 \\ 0 & S \end{pmatrix}$ (which are ρ -invariant and operate on our \hat{Q} in the same way), the additional conjugation by $\begin{pmatrix} R & 0 \\ 0 & R \end{pmatrix}$ or $\begin{pmatrix} S & 0 \\ 0 & S \end{pmatrix}$ in the definition of ψ on these groups, and the action of ρ on $(\mathbb{E}, \rho, \varepsilon)$ for the latter case. This proves the corollary. \square

We remark that in the second case considered in Corollary 11.12 the ambient spin group of the direct sum of 4 hyperbolic planes over \mathbb{E} comes, as seen in

Corollary 9.17, with three inequivalent maps onto the associated SO^1 group. The subgroup considered in Corollary 11.12 in this case is defined by the images in two of these representations (those which is not the defining representation as a spin group) being ρ -images of one another, after replacing one of them by an equivalent one. The fact that the double cover splits over this group, but it still maps to the defining representation with an order 2 kernel, is related to the fact that $-I$ equals its ρ -image. Indeed, of the order 2 elements in the kernels of the three representations, only $-I$ with ψ -image $-I$ satisfies the $\psi_{\hat{Q}} = \rho$ condition. The associated Gspin group maps onto the full special orthogonal group (with kernel \mathbb{F}^\times) in the first representation, but remains injective in the other two representations, allowing the bilinear form to be multiplied by scalars from \mathbb{F}^\times (the same scalar in both representations).

Every subspace of dimension 7 is contained in a space as we considered in Lemma 11.1. Indeed, just choose a non-zero vector norm from the space, and add a vector whose vector norm is the additive inverse of the chosen one. However, there is then a relation between the choice of the vector norm and the discriminant of the resulting 8-dimensional space, meaning that the description of the groups thus obtained depends on many choices. Hence we content ourselves with 7-dimensional spaces in which we can make the resulting space have discriminant 1, yielding a “canonical” complement, and do not pursue the rest (until a good description of non-isotropic 8-dimensional spaces becomes available).

12 Fields with Many Squares

Both the discriminant and the spinor norm take values in the group $\mathbb{F}^\times/(\mathbb{F}^\times)^2$. It is thus worthwhile to consider explicitly the cases where this group is very small. A field of characteristic different from 2 for which this group is trivial is called *quadratically closed*. This is the case, for example, where \mathbb{F} is algebraically closed, e.g., $\mathbb{F} = \mathbb{C}$. Over such a field \mathbb{F} (with characteristic different from 2), every two non-degenerate quadratic spaces of the same dimension n are isomorphic, hence the special orthogonal group can be denoted simply $SO(n, \mathbb{F})$. The group $SO^1(n, \mathbb{F})$ always coincides with $SO(n, \mathbb{F})$, as the spinor norm is trivial. Note that \mathbb{F} admits neither non-trivial quadratic field extensions nor non-split quaternion algebras, so that we may always take $\mathbb{E} = \mathbb{F} \times \mathbb{F}$ and B (or C) to be $M_2(\mathbb{F})$. Gathering the results of Theorems 3.4, 3.9, 4.4, 5.9, 6.4, 7.6, 9.16, 10.5, and 11.10 and their corollaries, we establish the following assertions:

$SO(1, \mathbb{F}) = \{1\}$, with $spin(1, \mathbb{F}) = \{\pm 1\}$ and $Gspin(1, \mathbb{F}) = \mathbb{F}^\times$.

$SO(2, \mathbb{F}) \cong \mathbb{F}^\times$, with $spin(2, \mathbb{F})$ being also \mathbb{F}^\times and $Gspin(2, \mathbb{F}) = \mathbb{F}^\times \times \mathbb{F}^\times$.

$spin(3, \mathbb{F}) = SL_2(\mathbb{F})$, $SO(3, \mathbb{F}) = PSL_2(\mathbb{F})$, and $Gspin(3, \mathbb{F}) = GL_2(\mathbb{F})$.

$spin(4, \mathbb{F}) = SL_2(\mathbb{F}) \times SL_2(\mathbb{F})$, $SO(4, \mathbb{F})$ is the quotient by $\{\pm(I, I)\}$, and $Gspin(4, \mathbb{F})$ is a subgroup $GL_2(\mathbb{F}) \times GL_2(\mathbb{F})$ determined by the equal determinant condition.

$spin(5, \mathbb{F}) = Sp_4(\mathbb{F})$, $SO(5, \mathbb{F}) = PSp_4(\mathbb{F})$, and $Gspin(5, \mathbb{F}) = GSp_4(\mathbb{F})$.

$spin(6, \mathbb{F}) = SL_4(\mathbb{F})$, $SO(6, \mathbb{F})$ is the quotient by $\pm I$ (this is not $PSL_4(\mathbb{F})$, since in order to obtain the latter group one must also divide by the two square roots of -1), and $Gspin(6, \mathbb{F}) = GL_4(\mathbb{F})$ since the $(\mathbb{F}^\times)^2$ condition on the determinant is vacuous over a quadratically closed field.

$spin(7, \mathbb{F})$ is the subgroup of those elements $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ of $SO\begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix} \subseteq GL_8(\mathbb{F})$ in which ad^t and bc^t are in \mathbb{F} and square to the determinants of the corresponding 4×4 blocks, and either bd^{-1} and ca^{-1} or ac^{-1} and db^{-1} are anti-symmetric and multiply to give minus their Pfaffian. $Gspin(7, \mathbb{F})$ is the group defined by the same conditions, but in which the bilinear form arising from $\begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}$ may be multiplied by a scalar.

Finally, $spin(8, \mathbb{F})$ has 3 inequivalent representations in which it maps onto the $SO\begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}$ with different order 2 kernels. $Gspin(8, \mathbb{F})$ maps to $SO\begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}$ with kernel \mathbb{F}^\times , but extending the two other representations of $spin(8, \mathbb{F})$ to it results in transformations which may multiply the bilinear form defined by $\begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}$ by non-trivial scalars.

We now present the case where $\mathbb{F}^\times/(\mathbb{F}^\times)^2$ has order 2 (and $ch\mathbb{F} \neq 2$). In this case there is a unique quadratic field extension of \mathbb{F} , which we denote \mathbb{E} . Hence all the (non-trivial) unitary groups defined over \mathbb{F} are based on \mathbb{E} with its Galois automorphism ρ over \mathbb{F} . One family of fields having this property is the family of finite fields of odd cardinality. As another example, recall that a field \mathbb{F} is *Euclidean* if it is ordered and every positive element is a square. For these fields we have

Proposition 12.1. *If \mathbb{F} is Euclidean then \mathbb{E} equals $\mathbb{F}(\sqrt{-1})$, it is quadratically closed, and we have $N_{\mathbb{F}}^{\mathbb{E}}(\mathbb{E}^\times) = (\mathbb{F}^\times)^2$. Moreover, the quaternion algebra $\left(\frac{-1, -1}{\mathbb{F}}\right)$, which we denote \mathbb{H} from now on, is not split and satisfies $N_{\mathbb{F}}^{\mathbb{H}}(\mathbb{H}^\times) = (\mathbb{F}^\times)^2$ as well.*

Proof. We know that $\mathbb{E} = \mathbb{F}(\sqrt{-1})$ since -1 cannot be a square in an ordered field. Thus, for any $z \in \mathbb{E}$, $N_{\mathbb{F}}^{\mathbb{E}}(z)$ may be presented as the sum of two squares in \mathbb{E} , not both zero if $z \neq 0$. The assertion $N_{\mathbb{F}}^{\mathbb{E}}(\mathbb{E}^\times) = (\mathbb{F}^\times)^2$ now follows from the properties of orderings. A similar argument shows that $N_{\mathbb{F}}^{\mathbb{H}}(\alpha)$ is the sum of four squares, hence non-zero for $\alpha \neq 0$. Hence \mathbb{H} is a division algebra and $N_{\mathbb{F}}^{\mathbb{H}}(\mathbb{H}^\times) = (\mathbb{F}^\times)^2$.

It remains to show that \mathbb{E} is quadratically closed. Any $z \in \mathbb{E}^\times$ can be written as r^2u with $r \in \mathbb{F}^\times$ and $u \in \mathbb{E}^1$: As $N_{\mathbb{F}}^{\mathbb{E}}(z) \in (\mathbb{F}^\times)^2$, it is the square of some $t \in \mathbb{F}^\times$, and by replacing t by $-t$ is necessary, we may assume $t > 0$. But then $t = r^2$, and $u = \frac{z}{r^2} \in \mathbb{E}^1$ as desired. But then Hilbert's Theorem 90 implies that $u = \frac{w^\rho}{w}$ for $w \in \mathbb{E}^\times$, and as this element is the quotient of $N_{\mathbb{F}}^{\mathbb{E}}(w) \in (\mathbb{F}^\times)^2$ and $w^2 \in (\mathbb{E}^\times)^2$, we find that u (hence also z) is a square in \mathbb{E} . Thus \mathbb{E} is indeed quadratically closed, which completes the proof of the proposition. \square

Note that \mathbb{H} and $M_2(\mathbb{F})$ are the only quaternion algebras over \mathbb{F} : For any symbol $\left(\frac{\alpha, \beta}{\mathbb{F}}\right)$ we may get an isomorphic quaternion algebra with α and β taken from $\{\pm 1\}$, yielding \mathbb{H} if $\alpha = \beta = -1$ and a split algebra otherwise. Thus, there are also two bi-quaternion algebras over \mathbb{F} , namely $M_2(\mathbb{H})$ and $M_4(\mathbb{F})$.

All these split over the quadratically closed extension \mathbb{E} . We also remark that the Artin–Schreier Theorem states that every field \mathbb{F} such that the algebraic closure of \mathbb{F} has non-trivial finite degree over \mathbb{F} (like \mathbb{R}) must be Euclidean.

For a non-degenerate quadratic form over a Euclidean field one defines the *signature*: An orthogonal basis can be normalized to have norms in ± 1 , and then the signature is the number of $+1$ s and the number of -1 s. These are two numbers (p, q) which sum to the dimension of the space, and we have

Proposition 12.2. *The signature classifies quadratic spaces over \mathbb{F} up to isometry.*

Proof. We can distinguish the spaces of signature $(n, 0)$ and $(0, n)$ (these spaces are called *definite*, the others *indefinite*) from the others by the fact that the first one has only positive vector norms, the second has only negative vector norms, and all the rest are isotropic. This completes the verification for dimensions 1 and 2. For a larger dimension, it suffices to prove that isotropic spaces with different signatures are not isometric. But each such space splits a hyperbolic plane, and the complements must have different signatures since a hyperbolic plane has signature $(1, 1)$. As these complements are not isometric by the induction hypothesis, the original spaces cannot be isometric by the Witt Cancellation Theorem. This proves the proposition. \square

For a finite field \mathbb{F} of odd cardinality, $\mathbb{F}^\times/(\mathbb{F}^\times)^2$ has order 2, but it is not Euclidean as it cannot be ordered (only fields of characteristic 0 may admit orderings). In fact, we have the following complement to Proposition 12.1:

Proposition 12.3. *Let \mathbb{F} be a field such that $\mathbb{F}^\times/(\mathbb{F}^\times)^2$ has order 2. Then if \mathbb{F} is not Euclidean then $N_{\mathbb{F}}^{\mathbb{E}}(\mathbb{E}^\times) = \mathbb{F}^\times$, \mathbb{E} is not quadratically closed, there are no non-split quaternion algebras over \mathbb{F} , and every quadratic form over \mathbb{F} is determined by its dimension and discriminant.*

Proof. The norm group $N_{\mathbb{F}}^{\mathbb{E}}(\mathbb{E}^\times)$ is a subgroup of \mathbb{F}^\times which contains $(\mathbb{F}^\times)^2$. Hence if the latter has index 2 in the former, $N_{\mathbb{F}}^{\mathbb{E}}(\mathbb{E}^\times)$ must coincide with one of these two groups. Now, the only quaternion algebra which may not split is $\mathbb{H} = \left(\frac{d, d}{\mathbb{F}}\right)$ where $d \in \mathbb{F}^\times$ is not a square. This algebra does split if and only if $d \in N_{\mathbb{F}}^{\mathbb{E}}(\mathbb{E}^\times)$, which is equivalent to $N_{\mathbb{F}}^{\mathbb{E}}(\mathbb{E}^\times)$ being the full group \mathbb{F}^\times by what we have seen above. \mathbb{E} cannot be quadratically closed in this case, since the norm of a square in \mathbb{E} is a square in \mathbb{F} .

We claim that if \mathbb{H} does not split then \mathbb{F} must be Euclidean. First observe that the product of the two generators of \mathbb{H} squares to $-d^2$, so that if \mathbb{H} does not split then $-1 \notin (\mathbb{F}^\times)^2$, and we may take $d = -1$. But then norms from \mathbb{E} are sums of two squares. It thus follows from the assumption $N_{\mathbb{F}}^{\mathbb{E}}(\mathbb{E}^\times) = (\mathbb{F}^\times)^2$ that $(\mathbb{F}^\times)^2$ is closed under addition, and as $\mathbb{F}^\times = (\mathbb{F}^\times)^2 \times \{\pm 1\}$ we find that $(\mathbb{F}^\times)^2$ defines an ordering on \mathbb{F} . This proves that \mathbb{F} is indeed Euclidean.

It remains to prove that in the non-Euclidean case, a quadratic form is determined by its dimension and discriminant. In dimension 1 the discriminant characterizes the quadratic space over any field. In dimension 2 it follows from

Lemma 3.1 that any space contains vectors of any given non-zero vector norm: For trivial discriminant this is clear (a hyperbolic plane), and for the non-trivial discriminant this follows from our assumption on $N_{\mathbb{F}}^{\mathbb{E}}(\mathbb{E}^{\times})$. The assertion for dimension 2 is now clear, by taking a vector of vector norm 1 and knowing what the orthogonal complement must be. Every quadratic space of dimension 3 (hence also larger) must therefore be isotropic: Fix an anisotropic vector v , the space v^{\perp} must contain some u with $|u|^2 = -|v|^2$, and then $u+v$ is isotropic. The assertion now follows by induction: If two quadratic spaces have the same dimension at least 3 and the same discriminant, then both are isotropic, both split hyperbolic planes, and in both the complement has the same dimension and the same discriminant. As the complements are isometric by the induction hypothesis, the same assertion holds for the original ones. This completes the proof of the proposition. \square

In addition to finite fields, every *quasi-finite* field (i.e., a perfect field which admits a unique extension of every finite order) of characteristic different from 2 satisfies the conditions of Proposition 12.3, and is not Euclidean. The same holds for (perfect) fields whose absolute Galois group misses some factors \mathbb{Z}_p for odd p in the pro-finite completion of \mathbb{Z} , such as direct limits of finite fields where the power of 2 in the exponent is bounded (otherwise the result is quadratically closed). Hence we call a non-Euclidean field of characteristic different from 2 such that $\mathbb{F}^{\times}/(\mathbb{F}^{\times})^2$ has order 2 *quadratically finite*.

In Proposition 12.1 we have seen that the quadratic extension of a Euclidean field is quadratically closed. As a complementary claim, Proposition 12.3 has the following

Corollary 12.4. *If \mathbb{F} is quadratically finite then so is its (unique) quadratic extension \mathbb{E} .*

Proof. First observe that if $z \in \mathbb{E}$ satisfies $N_{\mathbb{F}}^{\mathbb{E}}(z) \in (\mathbb{F}^{\times})^2$ then an argument similar to the last part of the proof of Proposition 12.1 shows that z can be presented as $\frac{tN_{\mathbb{F}}^{\mathbb{E}}(w)}{w^2}$ for some $t \in \mathbb{F}^{\times}$ and $w \in \mathbb{F}^{\times}$. It follows that $z \in (\mathbb{E}^{\times})^2$ since we clearly have $\mathbb{F}^{\times} \subseteq (\mathbb{E}^{\times})^2$. Thus the index of $(\mathbb{E}^{\times})^2$ in \mathbb{E}^{\times} can be at most 2, but it has to be precisely 2 since Proposition 12.3 shows that $\mathbb{E}^{\times}/(\mathbb{E}^{\times})^2$ cannot be of order 1. Since $-1 \in \mathbb{F}^{\times}$ is a square in \mathbb{E} , the latter field cannot be Euclidean, hence it is quadratically finite by Proposition 12.3. This proves the corollary. \square

For notational purposes, it will be convenient to identify the group $\mathbb{F}^{\times}/(\mathbb{F}^{\times})^2$, in both the Euclidean and quadratically finite cases, with $\{\pm 1\}$ (this is the Legendre symbol over q in the finite case). As the spinor norm takes values in this group, we write SO^+ for the SO^1 groups.

We know that $SO^+(V) = SO(V)$ whenever V has dimension 1 (and this group is trivial). But this is almost the only case where this may happen:

Proposition 12.5. *Let V be a quadratic space of dimension > 1 over a field \mathbb{F} which is either Euclidean or quadratically finite. Then $SO^+(V)$ has index 2*

in $SO(V)$ unless \mathbb{F} is Euclidean and the space is definite, a case in which the index is 1.

Proof. As $SO^+(V)$ is the kernel of a map $SO(V) \mapsto \{\pm 1\}$, it either coincides with $SO(V)$ (if the map is trivial) or has index 2 in it. Therefore it suffices to construct an element having non-trivial spinor norm, and prove that there is no such element in the exceptional case. Now, if \mathbb{F} is quadratically finite then the proof of Proposition 12.3 shows that there V contains vectors with vector norms in $(\mathbb{F}^\times)^2$ as well as anisotropic vectors whose vector norms do not lie in $(\mathbb{F}^\times)^2$. The same assertion clearly holds for indefinite spaces in the Euclidean case. Hence the composition of reflections in one vector of each vector norm yields the desired element of $SO(V)$. On the other hand, if \mathbb{F} is Euclidean and the space is definite then every reflection has the same vector norm. As Proposition 2.2 implies the elements of $SO(V)$ are products of an even number of reflections, the triviality of the spinor norm in this case follows. This proves the proposition. \square

Using Proposition 12.2, all the special orthogonal groups over a Euclidean field take the form $SO(p, q, \mathbb{F})$ for some natural numbers p and q , where we have $SO(p, q, \mathbb{F}) = SO(q, p, \mathbb{F})$ by a global sign inversion on the space. By Proposition 12.5, the subgroup $SO^+(p, q, \mathbb{F})$ has index 2 in $SO(p, q, \mathbb{F})$ unless $pq = 0$. On the other hand, it follows from Proposition 12.3 that a special orthogonal group over a quadratically finite field takes the form $SO(n, \varepsilon, \mathbb{F})$, where n is the dimension and $\varepsilon \in \{\pm\}$ represents the discriminant. Moreover, as rescaling may change the discriminant in odd dimensions, we write just $SO(n, \mathbb{F})$ for odd n . Proposition 12.5 shows that $SO^+(n, \varepsilon, \mathbb{F})$ or $SO^+(n, \mathbb{F})$ always has index 2 there if $n > 1$ (and otherwise the groups are trivial). The spin and Gspin groups are denoted with SO or SO^+ replaced by $spin$ and $Gspin$ respectively. For the finite fields \mathbb{F}_q , where we may replace \mathbb{F}_q by simply q in the notation, this means that $spin(n, \varepsilon, q)$ and $SO(n, \varepsilon, q)$ have the same cardinality (for $n > 1$), while the cardinality of $SO^+(n, \varepsilon, q)$ is half that number and to get the cardinality of $Gspin(n, \varepsilon, q)$ we multiply by $q - 1$.

Recall that in some settings we write results in terms of unitary groups. Arguments which are similar to the orthogonal groups over quadratically closed and Euclidean fields show that unitary spaces over quadratically finite fields are determined by their dimensions, while over Euclidean field they have signatures just like the quadratic ones. Hence we shall use notations like $U_{\mathbb{E}, \rho}(n)$ in the former case and $U_{\mathbb{E}, \rho}(p, q)$ in the latter case, as well as U replaced by SU , GU , or GSU when required. The groups $Sp_{\mathbb{H}}(p, q)$ are defined in a similar manner in the Euclidean case.

When we apply Theorems 3.4, 3.9, 4.4, 5.9, 6.4, 7.6, 9.16, 10.5, and 11.10, and their corollaries, to the quadratically finite field case, the results we get are as follows:

$SO(1, \mathbb{F})$ as well as $SO^+(1, \mathbb{F})$ are $\{1\}$, while $spin(1, \mathbb{F})$ equals $\{\pm 1\}$ and $Gspin(1, \mathbb{F})$ is \mathbb{F}^\times .

$SO(2, +, \mathbb{F})$ and $spin(2, +, \mathbb{F})$ are both isomorphic to \mathbb{F}^\times , $SO^+(2, +, \mathbb{F})$ is $(\mathbb{F}^\times)^2$, and $Gspin(2, +, \mathbb{F}) = \mathbb{F}^\times \times \mathbb{F}^\times$. On the other hand, both $SO(2, -, \mathbb{F})$ and $Spin(2, -, \mathbb{F})$ are isomorphic to \mathbb{E}^1 (or equivalently $U_{\mathbb{E}, \rho}(1)$), $SO^+(2, -, \mathbb{F})$ is $(\mathbb{E}^1)^2$, and $Gspin(2, -, \mathbb{F})$ equals \mathbb{E}^\times (or $GU_{\mathbb{E}, \rho}(1)$).

$spin(3, \mathbb{F}) = SL_2(\mathbb{F}) = SU_{\mathbb{E}, \rho}(2)$, $Gspin(3, \mathbb{F}) = GL_2(\mathbb{F}) = GSU_{\mathbb{E}, \rho}(2)$, $SO^+(3, \mathbb{F}) = PSL_2(\mathbb{F})$, and $SO(3, \mathbb{F}) = PGL_2(\mathbb{F})$.

$spin(4, +, \mathbb{F})$ is $SL_2(\mathbb{F}) \times SL_2(\mathbb{F})$, $Gspin(4, +, \mathbb{F})$ is a subgroup of the product $GL_2(\mathbb{F}) \times GL_2(\mathbb{F})$ determined by the equal determinant condition, and $SO^+(4, +, \mathbb{F})$ and $SO(4, +, \mathbb{F})$ are the appropriate quotients. For the non-trivial discriminant, we get $SL_2(\mathbb{E})$ for $spin(4, -, \mathbb{F})$ and $PSL_2(\mathbb{E})$ for $SO^+(4, -, \mathbb{F})$, while $SO(4, -, \mathbb{F})$ is obtained from the latter group as a direct product with $\{\pm 1\}$ and $Gspin(4, -, \mathbb{F}) = GL_2^{\mathbb{E}^\times}(\mathbb{E})$.

The group $spin(5, \mathbb{F})$ is $Sp_4(\mathbb{F})$, $SO^+(5, \mathbb{F})$ is $PSp_4(\mathbb{F})$, $Gspin(5, \mathbb{F})$ equals $GSp_4(\mathbb{F})$, and $SO(5, \mathbb{F})$ is $PGSp_4(\mathbb{F})$.

$spin(6, +, \mathbb{F}) = SL_4(\mathbb{F})$, $SO^+(6, +, \mathbb{F})$ equals $PSL_4(\mathbb{F})$ if $-1 \notin (\mathbb{F}^\times)^2$ (but not otherwise!), $SO(6, +, \mathbb{F})$ is the direct product with $\{\pm 1\}$, and $Gspin(6, +, \mathbb{F})$ equals $GL_4^{(\mathbb{F}^\times)^2}(\mathbb{F})$. With the other discriminant we get the groups $SU_{\mathbb{E}, \rho}(4)$ for $spin(6, -, \mathbb{F})$ and $GSU_{\mathbb{E}, \rho}(4)$ for $Gspin(6, -, \mathbb{F})$, with the groups $SO^+(6, -, \mathbb{F})$ and $SO(6, -, \mathbb{F})$ being the appropriate quotients.

$spin(7, \mathbb{F})$ may again be described as the subgroup of $SO^+\begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix} \subseteq GL_8(\mathbb{F})$ in which the elements, presented as block matrices $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$, satisfy the conditions that ad^t and bc^t are scalars (with squares some block determinants) and bd^{-1} and ca^{-1} , or ac^{-1} and db^{-1} , are anti-symmetric and related to one another via the Pfaffian being their product. For $Gspin(7, \mathbb{F})$ we relax the $SO\begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}$ condition to allow scalar multiplication of the underlying bilinear form.

The group $spin(8, +, \mathbb{F})$ admits 3 inequivalent representations as double covers of $SO^+\begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}$ groups. These representations are restrictions of representations of $Gspin(8, +, \mathbb{F})$, in one of which the kernel becomes \mathbb{F}^\times and in the other two the bilinear form may be multiplied by scalars. $spin(8, -, \mathbb{F})$ is the subgroup of $spin(8, +, \mathbb{E})$ (which has the structure we just considered by Corollary 12.4, whence also the notation) in which two of the representations to $SO^+\begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}$ (over \mathbb{E}) become isomorphic, with ρ and conjugating by $\begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}$ being the isomorphism. $Gspin(8, -, \mathbb{F})$ is defined by the same condition on $Gspin(8, +, \mathbb{E})$, with the two representations which become isomorphic being those in which the bilinear form is multiplied by scalars (which are only from \mathbb{F}^\times in this subgroup).

When we consider the case of Euclidean fields, recall that $(\mathbb{F}^\times)^2$ is the set of positive elements. Hence we denote it \mathbb{F}_+^\times , and furthermore replace any such superscript by simply $+$. Recall that the determinant of a space of signature (p, q) is $(-1)^q$, but for the discriminant, which matters to us only for even n , we must multiply by $(-1)^{n/2}$. Note that double covers which are based on choosing a square root split here, since we have the canonical choice of the positive square root. The results of Theorems 3.4, 3.9, 4.4, 5.9, 6.4, 7.6, 9.16, 10.5, and 11.10 then take the form appearing below:

$SO(1, 0, \mathbb{F}) = SO^+(1, 0, \mathbb{F}) = \{1\}$, $Spin(1, 0, \mathbb{F}) = \{\pm 1\}$, and $GSpin(1, 0, \mathbb{F})$ equals \mathbb{F}^\times .

$SO(2, 0, \mathbb{F}) = SO^+(2, 0, \mathbb{F})$ as well as $spin(2, 0, \mathbb{F})$ are $\mathbb{E}^1 = U_{\mathbb{E}, \rho}(1, 0)$ and $GSpin(2, 0, \mathbb{F}) = \mathbb{E}^\times = GU_{\mathbb{E}, \rho}(1, 0)$. On the other hand, $SO(1, 1, \mathbb{F}) = \mathbb{F}^\times$, $SO^+(1, 1, \mathbb{F}) = \mathbb{F}_+^\times$, $Spin(1, 1, \mathbb{F}) = \mathbb{F}^\times$ as well, and $GSpin(1, 1, \mathbb{F}) = \mathbb{F}^\times \times \mathbb{F}^\times$.

$spin(3, 0, \mathbb{F})$ is \mathbb{H}^1 or equivalently $SU_{\mathbb{E}, \rho}(2, 0)$, $SO(3, 0, \mathbb{F}) = SO^+(3, 0, \mathbb{F})$ is obtained as the quotient by $\{\pm 1\}$, and $Gspin(3, 0, \mathbb{F}) = \mathbb{H}^\times = GSU_{\mathbb{E}, \rho}(2, 0)$. On the other hand, $spin(1, 2, \mathbb{F})$ is $SL_2(\mathbb{F})$ (or equivalently $SU_{\mathbb{E}, \rho}(1, 1)$) hence $SO^+(1, 2, \mathbb{F}) = PSL_2(\mathbb{F})$, $GSpin(1, 2, \mathbb{F})$ equals $GL_2(\mathbb{F})$ (or $GSU_{\mathbb{E}, \rho}(1, 1)$), and $SO^+(1, 2, \mathbb{F})$ is $PGL_2(\mathbb{F})$.

The group $spin(4, 0, \mathbb{F})$ is $\mathbb{H}^1 \times \mathbb{H}^1$ or its isomorph $SU_{\mathbb{E}, \rho}(2, 0) \times SU_{\mathbb{E}, \rho}(2, 0)$, $Gspin(4, 0, \mathbb{F})$ is the subgroup of $\mathbb{H}^\times \times \mathbb{H}^\times$ consisting of pairs of quaternions with the same norm, and $SO^+(4, 0, \mathbb{F}) = SO(4, 0, \mathbb{F})$ is the corresponding quotient. Similarly, but with the split algebra, $spin(2, 2, \mathbb{F})$ is $SL_2(\mathbb{F}) \times SL_2(\mathbb{F})$ (or equivalently $SU_{\mathbb{E}, \rho}(1, 1) \times SU_{\mathbb{E}, \rho}(1, 1)$), $GSpin(2, 2, \mathbb{F})$ is the “same determinant subgroup” of $GL_2(\mathbb{F}) \times GL_2(\mathbb{F})$, and $SO^+(2, 2, \mathbb{F})$ and $SO(2, 2, \mathbb{F})$ are the appropriate quotients. On the other hand, $spin(1, 3, \mathbb{F}) = SL_2(\mathbb{E})$, $SO^+(1, 3, \mathbb{F})$ is $PSL_2(\mathbb{E})$, $SO^+(1, 3, \mathbb{F})$ is the direct product of the latter group with $\{\pm 1\}$, and $GSpin(1, 3, \mathbb{F})$ equals $GL_2^{\mathbb{F}^\times}(\mathbb{E})$.

We also have $spin(5, 0, \mathbb{F}) = Sp_{\mathbb{H}}(2, 0)$ and $Gspin(5, 0, \mathbb{F}) = GSp_{\mathbb{H}}(2, 0)$, with $SO^+(5, 0, \mathbb{F}) = SO(5, 0, \mathbb{F})$ being the quotient $GSp_{\mathbb{H}}(2, 0)$ of the former modulo $\{\pm 1\}$ or the latter modulo \mathbb{F}^\times . In a similar manner, $spin(4, 1, \mathbb{F})$ is $Sp_{\mathbb{H}}(1, 1)$, $Gspin(4, 1, \mathbb{F})$ is $GSp_{\mathbb{H}}(1, 1)$, and $SO^+(4, 1, \mathbb{F})$ and $SO(4, 1, \mathbb{F})$ are the usual quotients. In addition, $Sp_4(\mathbb{F})$ is $spin(2, 3, \mathbb{F})$ so that $SO^+(2, 3, \mathbb{F})$ is $PSp_4(\mathbb{F})$, and $GSpin(2, 3, \mathbb{F})$ equals $GSp_4(\mathbb{F})$.

The group $spin(5, 1, \mathbb{F})$ is $GL_2^1(\mathbb{H})$, $GSpin(5, 1, \mathbb{F})$ equals $GL_2(\mathbb{H}) \times \{\pm 1\}$ (the double cover splits, and the superscript $+$ is unnecessary by Lemma 6.6 and the fact that $N_{\mathbb{F}}^{\mathbb{H}}(\mathbb{H}^\times) = (\mathbb{F}^\times)^2$), and $SO^+(5, 1, \mathbb{F})$ and $SO(5, 1, \mathbb{F})$ are the quotients (the latter being the direct product of the former with $\{\pm 1\}$). Using the split algebra, $Spin(3, 3, \mathbb{F})$ is just $SL_4(\mathbb{F})$, $SO^+(3, 3, \mathbb{F})$ is $PSL_4(\mathbb{F})$ as $-1 \notin (\mathbb{F}^\times)^2$, $GSpin(3, 3, \mathbb{F})$ is isomorphic to $GL_4^+(\mathbb{F}) \times \{\pm 1\}$ (a split double cover), and $SO(3, 3, \mathbb{F})$ equals $PSL_4(\mathbb{F}) \times \{\pm 1\}$. $spin(6, 0, \mathbb{F})$ equals $SU_{\mathbb{E}, \rho}(4, 0)$, $GSpin(6, 0, \mathbb{F})$ is $GSU_{\mathbb{E}, \rho}(4, 0)$, and $SO^+(6, 0, \mathbb{F}) = SO(6, 0, \mathbb{F})$ is obtained as both the appropriate quotients. Finally, $spin(4, 2, \mathbb{F})$ is isomorphic to $SU_{\mathbb{E}}(2, 2)$, $GSpin(4, 2, \mathbb{F})$ is $GSU_{\mathbb{E}}(2, 2)$, and the usual quotients give $SO^+(4, 2, \mathbb{F})$ and $SO(4, 2, \mathbb{F})$.

The group $spin(4, 3, \mathbb{F})$ is the subgroup of $SO^+\left(\begin{smallmatrix} 0 & I \\ I & 0 \end{smallmatrix}\right)$ (8×8 matrices) consisting of those block matrices $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ in which ad^t and bc^t are in \mathbb{F} and square to $\det a = \det d$ and $\det b = \det c$ respectively, and where either bd^{-1} and ca^{-1} or ac^{-1} and db^{-1} belong to $M_4^{as}(\mathbb{F})$ and multiply to minus their common Pfaffian. $Gspin(4, 3, \mathbb{F})$ is described by the same condition on the group of matrices in $GL_8(\mathbb{F})$ whose action multiplies $\begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}$ by a scalar (with some extra condition extending the SO^1 condition). For $Gspin(5, 2, \mathbb{F})$ we get the subgroup of $GSp_4(\mathbb{H})$, elements $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ of which satisfy the conditions that $a\bar{d}^t$ and $b\bar{c}^t$ are scalars squaring to $N_{\mathbb{F}}^{M_2(B)}(a) = N_{\mathbb{F}}^{M_2(B)}(d)$ and $N_{\mathbb{F}}^{M_2(B)}(b) = N_{\mathbb{F}}^{M_2(B)}(c)$ respectively, and either the pair bd^{-1} and ca^{-1} or the pair ac^{-1} and db^{-1} is a pair of matrices in $M_2^{Her}(B)$ which are minus the adjoints of one another. $spin(5, 2, \mathbb{F})$ is the

group of the matrices in $Sp_4(\mathbb{H})$ having these properties. $Gspin(6, 1, \mathbb{F})$ and $spin(6, 1, \mathbb{F})$ are similar subgroups of $GSp_4(\mathbb{H})$ and $Sp_4(\mathbb{H})$, but in which the pairs of minus adjoint matrices are bd^{-1} and $-ca^{-1}$ or ac^{-1} and $-db^{-1}$.

$spin(6, 2, \mathbb{F})$ and $Gspin(6, 2, \mathbb{F})$ are double covers of $Sp_4(\mathbb{H})$ and $GSp_4(\mathbb{H})$ respectively in two, inequivalent ways. We have omitted the superscript \mathbb{F}^2 since the reduced norms from \mathbb{H} , hence also from $M_2(\mathbb{H})$ by Lemma 6.6, are non-negative, whence the map φ from Proposition 9.8 is trivial in this case. $spin(4, 4, \mathbb{F})$ maps in 3 inequivalent ways to $SO^+(\begin{smallmatrix} 0 & I \\ I & 0 \end{smallmatrix})$ with kernels of order 2, and $Gspin(4, 4, \mathbb{F})$ maps in one representation to $SO(\begin{smallmatrix} 0 & I \\ I & 0 \end{smallmatrix})$ with kernel \mathbb{F}^\times but multiplies the bilinear form by arbitrary scalars in the other two representations (where its kernel remains $\{\pm 1\}$). The group $spin(7, 1, \mathbb{F})$ is defined by the condition on $spin(8, \mathbb{E})$ (recall that \mathbb{E} is quadratically closed by Proposition 12.1) which states that conjugating one representations to the group $SO^+(\begin{smallmatrix} 0 & I \\ I & 0 \end{smallmatrix})$ over \mathbb{E} by $(\begin{smallmatrix} I & 0 \\ 0 & -I \end{smallmatrix})$ yields the ρ -image of the other representation. For $Gspin(7, 1, \mathbb{F})$ we apply the same condition on $Gspin(8, \mathbb{E})$ using two representations in which the operation multiplies the bilinear form by scalars (from \mathbb{F}^\times here). The groups $spin(5, 3, \mathbb{F})$ and $Gspin(5, 3, \mathbb{F})$ are obtained in the same manner but with each 4×4 identity matrix replaced by $(\begin{smallmatrix} I & 0 \\ 0 & -I \end{smallmatrix})$ (involving 2×2 identity matrices).

We remark that the case of the spin group in signature $(6, 2)$ over $\mathbb{F} = \mathbb{R}$ was considered in [SH], using Clifford algebras, Eichler transformations, and some real, complex, and quaternionic analytic tools. The homomorphism denoted ϕ in Lemma 6.10 of that reference is just $a \mapsto t\bar{a}^{-1}$ on $GL_2(\mathbb{H}) = GL_2^{(\mathbb{F}^\times)^2}(\mathbb{H})$, with the square roots being positive. Proposition 6.11 there is the projectivization of our Proposition 9.10. Now, the notion of positive definiteness extends from $\mathbb{F} = \mathbb{R}$ to any Euclidean field. It thus seems reasonable that the action of $Sp_4(\mathbb{H})$ on the subset of $M_2^{Her}(\mathbb{H}) \otimes \mathbb{E}$ in which the “imaginary part” (which is also well-defined) is positive definite, as well as the ψ -images of the elements of $\widetilde{Sp}_4(\mathbb{H})$ lying over $g \in Sp_4(\mathbb{H})$ being those which send Z^t (for Z in the latter space) to $g(Z)^t$, also extend to this more general setting. However, as we have seen, these aspects of the theory are not required for obtaining the general result.

We do not get presentations of the definite spin groups $spin(7, 0, \mathbb{F})$ and $spin(8, 0, \mathbb{F})$ here, since a definite space of dimension 7 does not represent its discriminant, and a definite space of dimension 8 is not isotropic, and our methods for spaces of dimensions 7 and 8 require these properties.

Unitary groups preserving sesqui-linear forms of dimension 3 do not arise in the context of orthogonal groups since the dimension 8 of such special unitary groups is not the dimension of any orthogonal group. Sesqui-linear forms of signature $(3, 1)$ also do not appear here because of the discriminant 1 condition. For $\mathbb{F} = \mathbb{R}$ we can also derive this fact from Lie theory: The dimension of $SU_{\mathbb{C}}(3, 1)$ is indeed 15, but its maximal compact subgroup $S(U(3) \times U(1))$ has dimension 9, which does not equal the dimension of $SO(p) \times SO(q)$ for any pair (p, q) with sum 6 (the required dimension is $\frac{p(p-1)}{2} + \frac{q(q-1)}{2}$, which attains only the values 15, 10, 7, and 6).

The splitting of the double covers here comes from the splitting of the se-

quence

$$1 \rightarrow \{\pm 1\} \rightarrow \mathbb{F}^\times \rightarrow (\mathbb{F}^\times)^2 \rightarrow 1.$$

This happens wherever the Abelian group \mathbb{F}^\times contains $\{\pm 1\}$ as a direct summand (e.g., when \mathbb{F} may be ordered, when $(\mathbb{F}^\times)^2$ is free like in number fields of class number 1 with no complex roots of unity, etc.). Note that when the double cover $\tilde{A}^{(\mathbb{F}^\times)^2}$ splits, the elements $(g, |g|^2)$ with $g \in A^- \cap A^\times$ will never lie all in the splitting group, as they generate the full double cover by Theorem 5.9.

On the other hand, for quadratically closed and Euclidean fields the sequence

$$1 \rightarrow (\mathbb{F}^\times)^2 \rightarrow \mathbb{F}^\times \rightarrow \mathbb{F}^\times/(\mathbb{F}^\times)^2 \rightarrow 1$$

splits. This fact is related to the full special orthogonal group admitting a double cover inside the Gspin group: For the quadratically closed case as well as the definite case this is just the usual spin group. In the indefinite case the double cover is obtained by imposing the condition that a certain reduced norm, determinant, or multiplier takes only the values ± 1 . In fact, one can show that the only additional case where this sequence splits is for a quadratically finite field in which $-1 \notin (\mathbb{F}^\times)^2$ (for the finite field \mathbb{F}_q this happens if and only if $q \equiv 3 \pmod{4}$). The double covers in this case are obtained in a way similar to the indefinite case.

We remark that the cardinality of the group $\mathbb{F}^\times/(\mathbb{F}^\times)^2$ can be either infinite or any finite power of 2. To see this, observe that if $\mathbb{K} = \mathbb{F}((X))$ then $\mathbb{K}^\times/(\mathbb{K}^\times)^2$ is generated by the injective image of $\mathbb{F}^\times/(\mathbb{F}^\times)^2$ and the class of X (another element of order 2 in $\mathbb{K}^\times/(\mathbb{K}^\times)^2$ which is independent of $\mathbb{F}^\times/(\mathbb{F}^\times)^2$). For the case the group has order 4 we again have different types of fields. Indeed, if \mathbb{K} is our field and it takes the form $\mathbb{F}((X))$ then if \mathbb{F} is quadratically finite then \mathbb{K} admits only one non-split quaternion algebra (this is also the case for the p -adic numbers for an odd prime p or their finite extensions), while for Euclidean \mathbb{F} there are 3 non-isomorphic quaternion algebras over \mathbb{K} which are not split. The description of the quadratic spaces over these fields will thus be different, with the first case probably resembling the results appearing in Section 2 of Chapter IV of [S]. We leave these questions, as well as the question whether every field \mathbb{F} in which $\mathbb{F}^\times/(\mathbb{F}^\times)^2$ has order 4 resembles one of these families, for future research.

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